March 2, 2015

1. If two solutions of $x^3 + px + q = 0$ are $x_1 = 1$ and $x_2 = 2$ then q equals

   (A) 3 (B) -3 (C) 6 (D) -6 (E) 2

2. How many four-digit integers are there with no two consecutive zeros?

   (A) 8919 (B) 8829 (C) 8001 (D) 4232 (E) 4156

3. A regular octahedral die has faces numbered 1 though 8. It is rolled four times obtaining a sum S satisfying $4 \leq S \leq 32$. What is the probability that $S(36 - S) < 180$?

   (A) 0 (B) 1/512 (C) 3/1024 (D) 5/2048 (E) 7/4096

4. What is the sum of the coordinates of the point (x, y) that satisfies both $x^2 + (y-5)^2 = 4$ and $(x - 12)^2 + y^2 = 121$?

   (A) 75/13 (B) 77/13 (C) 79/13 (D) 81/13 (E) 83/13

5. A teacher can grade 12 tests per hour by herself. Working with an assistant, she can grade 20 tests per hour, but it takes two hours to train the assistant. Let N be the least number of tests that can be graded faster by training an assistant. Then N belongs to the following interval:

   (A) $60 - 79$ (B) $80 - 119$ (C) $120 - 239$ (D) $240 - 244$ (E) $245 - 299$

6. What is the greatest integer k such that 3^k divides 100!

   (A) 33 (B) 44 (C) 47 (D) 48 (E) 50

7. What is the remainder when the number $N = 1^2 + 3^2 + 5^2 + \cdots 99^2$ is divided by 100?

   (A) 0 (B) 50 (C) 60 (D) 70 (E) 80

8. What is the largest value of n such that division of each of the numbers $1621, 2237$, and 2545 by n leave the same remainder?

   (A) 7 (B) 11 (C) 33 (D) 77 (E) 308

9. For how many integers k is it true that $\frac{k^3+8}{k^2-4}$ is an integer?

   (A) 2 (B) 3 (C) 4 (D) 5 (E) 6

10. The curve $y = 9x^2 + 8x + |c|$ has at least one real zero for all c such that $a \leq c \leq b$. What is $b - a$?

    (A) 16/9 (B) 24/9 (C) 32/9 (D) 49/9 (E) 51/9

11. The curve $(|y|-x^2)(|x|-y^2) = 0$ divides the plane into some bounded and some unbounded regions. How many such regions are there?

    (A) 4 (B) 6 (C) 8 (D) 12 (E) 16

12. Find the last digit of $1 + 3 + 3^2 + \cdots + 3^{2015}$.

    (A) 3 (B) 9 (C) 7 (D) 0 (E) 1

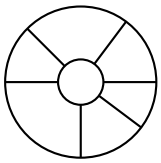
13. Evaluate $\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\cdots\left(1 - \frac{1}{2015^2}\right)$.

    (A) $\frac{1006}{2015}$ (B) $\frac{1007}{2015}$ (C) $\frac{1008}{2015}$ (D) $\frac{1009}{2015}$ (E) $\frac{202}{403}$

14. A restaurant makes two strengths of iced tea. They start with one container of pure water and another of tea concentrate. The first step of mixing is to pour enough liquid from container A into container B to double the volume in container B. The contents of container B are thoroughly mixed and then enough liquid in container B is poured into container A in order to double the volume that was left in container A. After this mixing, the two containers have the same volume and the blend in container A consists of g gallons of pure water and $g + 3$ gallons of tea concentrate. How many gallons of (the original) tea concentrate are in container A after the second mixing?

    (A) 7/2 (B) 9/2 (C) 11/2 (D) 13/2 (E) 15/2

15. A round sign is divided into seven sections as in the diagram. The sign is to be painted with five colors (blue, green, orange, red and yellow) so that each section is painted with one color and no adjacent sections are the same color. Also, each color must be used at least once and no color can be used more than twice. In how many different ways can the sign be painted?



    (A) 1260 (B) 1480 (C) 2160 (D) 2520 (E) 5040

16. Al wants to buy a new TV and the one he wants just went on sale for 25% off. Alas, Al forgot about the 8% sales tax and so when the TV is rung up, he finds he is \$35 short. The store manager comes by and says, "Give it to him for 30% off." With that Al can buy the TV and so he walks out with the TV and \$5.50 in his pocket. How much is the sales tax on Al's purchase?

    (A) \$37.50 (B) \$40.50 (C) \$41.25 (D) \$42.00 (E) \$43.74

17. For real numbers $a > b > 0$, let $P = (a, c)$ and $Q = (b, d)$ be points on the parabola given by $y = x^2$ and let $R = (a, f)$ and $S = (b, g)$ be points on the parabola given by $y = 4x^2$. If the segment $\overline{RS}$ is twice as long as $\overline{PQ}$, then what is the value of $a + b$?

    (A) 1/4 (B) 1/2 (C) 1 (D) 2 (E) 4

18. The vertical line L given by $x = -2$ and the point $F = (4, 3)$ determine a unique parabola in the plane. This parabola consists of the set of points $R = (f, g)$ where the distance from R to F is the same as the distance from R to the point $T = (-2, g)$ on the line L. In general, the line through R that is tangent to the parabola at R (it touches the parabola at R but does not cross over) is perpendicular to the line $\overleftrightarrow{FT}$. If the slope of $\overleftrightarrow{FT}$ is 1/2, what is the x-coordinate of R?

    (A) 1 (B) 3/2 (C) 7/4 (D) 9/4 (E) 5/2

19. At the beginning of the year there were 25 students in class, an odd number of them was female. When 7 new students joined the class, the proportion of female students increased by 25%. How many female students are there in the class now?

    (A) 6 (B) 7 (C) 8 (D) 12 (E) 16

20. What is the value of the expression $\frac{1}{2!}+\frac{2}{3!}+\ldots+\frac{2014}{2015!}$? ($n!$ is the product of $1\cdot 2\cdot \ldots \cdot n$)

    (A) $1-\frac{1}{2015!}$ (B) $1+\frac{1}{2015!}$ (C) $1-\frac{2014}{2015!}$ (D) $1+\frac{2014}{2015!}$ (E) $1-\frac{2014!}{2015!}$

21. The odometer on an automobile gives six-digit readings, one for each mileage from $000,000$ to $299,999$. A reading is "palindromic" if it reads the same from left to right and from right to left, for example $027,720$ and $163,361$. Find the total number of palindromic readings.

    (A) 100 (B) 200 (C) 300 (D) 400 (E) 500

22. How many ways are there to pay one dollar, using nickels, dimes and quarters?

    (A) 26 (B) 29 (C) 30 (D) 31 (E) 32

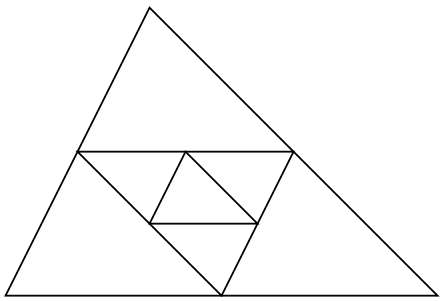
23. Which of the following numbers can be obtained by taking the difference of a 3-digit decimal number $n = abc_{10}$ with the number $n' = cba_{10}$, obtained by writing the digits of n in reverse order?

    (A) 9 (B) 97 (C) 297 (D) 400 (E) 496

24. Cards are turned over one at a time from a well shuffled 52 card deck until the first heart appears. The probability that exactly 5 cards are required is in which of the following intervals?

    (A) $[0, .2]$ (B) $(.2, 4]$ (C) $(.4, .6]$ (D) $(.6, .8]$ (E) $(.8, 1]$

25. There are three triangles of different sizes: small, medium and large. The small one is inscribed in the medium one such that its vertices are at the midpoints of the three edges of the medium one. The medium triangle is inscribed in the large triangle in the same way as shown in the figure. If the small triangle has area 1, what is the sum of the areas of the three triangles?



    (A) 14 (B) 16 (C) 19 (D) 21 (E) 25