UNC Charlotte 2010 Algebra

with solutions

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1. Let $y = mx + b$ be the image when the line $x - 3y + 11 = 0$ is reflected across the x-axis. The value of $m + b$ is:

   (A) -6 **(B)** -5 **(C)** -4 **(D)** -3 **(E)** -2

   Solution: C. The slope of the given line is $1/3$ with y-intercept $(0, 11/3)$. The reflected line has slope $-1/3$ and y-intercept $(0, -11/3)$. Then $m + b = -1/3 + (-11/3) = -4$.

2. Which of the following equations have the same graphs?

   I. $y = x - 2$ II. $y = (x^2 - 4)/(x + 2)$ III. $(x + 2)y = x^2 - 4$

   (A) I. and II. only **(B)** I. and III. only **(C)** II. and III. only

   (D) I., II., and III. **(E)** None. All the equations have different graphs.

   Solution: E. Note that when $x = -2$, graph I contains only the point $y = -4$, graph II contains no points with $x = -2$, and graph III contains the entire vertical line $x = -2$.

3. When 97^3 is expressed in base 100 notation, what is the sum of the 'digits'? (The digits are the coefficients of 100^2, 100, and 1.)

   (A) 88 **(B)** 143 **(C)** 162 **(D)** 187 **(E)** 190

   Solution: E. Expand to get $97^3 = (100-3)^3 = 100^3 - 3 \cdot 100^2 \cdot 3 + 3 \cdot 100 \cdot 3^2 - 3^3 = 100^2 \cdot (91) + 27 \cdot 100 - 27 = 91 \cdot 100^2 + 26 \cdot 100 + 73$, so the digits are 91, 26, and 73 and the sum is 190.

4. Find the maximum value of $f(x) = \sqrt{x^2 + 22x + 121} + \sqrt{x^2 - 26x + 169}$ over the interval $-12 \le x \le 12$.

   (A) 24 **(B)** 26 **(C)** 28 **(D)** 30 **(E)** 32

   Solution: B. Note that $f(x) = |x + 11| + |x - 13|$, whose maximum value over $[-12, 12]$ occurs at (-12); $f(-12) = 26$.

5. How many triples of positive integers (x, y, z) satisfy both

   - $x \leq y \leq z$ and
   - $x^2 + y^2 + z^2 = 4(x + y + z) + 38$.

   (A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** more than 3

   Solution: E. Add 12 to both sides of the second equation to get $x^2 - 4x + 4 + y^2 - 4y + 4 + z^2 - 4z + 4 = 50$. The left side is just $(x-2)^2 + (y-2)^2 + (z-2)^2$, and a case by case examination reveals four solutions, $(5, 6, 7)$, $(2, 3, 9)$, $(2, 7, 7)$, and $(1, 2, 9)$.

6. The number of positive integers k for which the equation $kx + 12 = 3k$ has an integer solution for x is

   (A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

   Solution: D. We have $x = 3 - \frac{12}{k}$, so k must be a positive integer that divides 12 and each such choice will have an integer solution for x. So $k = 1, 2, 3, 4, 6, 12$ are the values of k.

7. For all non-zero real numbers x and y such that $x - y = xy$, the value of $1/x - 1/y$ is

   (A) $1/2$ **(B)** 0 **(C)** 1 **(D)** 2 **(E)** -1

   Solution: E. Dividing both sides of $x - y = xy$ by $-xy$ yields $1/x - 1/y = -1$.

8. The sum of 67 consecutive integers is 2010. What is the least of these numbers?

   (A) -3 **(B)** -1 **(C)** 1 **(D)** 3 **(E)** 4

   Solution: A. We have $x + (x+1) + (x+2) + \ldots + (x+66) = 67x + (1 + 2 + 3 + \ldots + 66) = 67x + 66(67)/2 = 2010$. We find that $x = -3$.

9. Suppose $f(0) = 3$ and $f(n) = f(n-1) + 2$. Let $T = f(f(f(f(5))))$. What is the sum of the digits of T?

   (A) 6 **(B)** 7 **(C)** 8 **(D)** 9 **(E)** 10

   Solution: C. In fact, $f(n) = 2n + 3$, and $T = f(f(f(f(5)))) = f(f(f(13))) = f(f(29)) = f(61) = 125$.

10. Find the volume of a rectangular box whose left side, front side, and bottom have areas of 12 square inches, 30 square inches and 90 square inches, respectively.

    (A) 240 cubic inches **(B)** 300 cubic inches **(C)** 120 cubic inches

    (D) 180 cubic inches **(E)** 360 cubic inches

    Solution: D. Let w, h, and l denote the three dimensions. Then the given information can be written $wh = 12$, $wl = 30$, and $lh = 90$. Taking products of all three gives $w^2h^2l^2 = 12 \times 30 \times 90$. It follows that Volume $= whl = \sqrt{12 \cdot 30 \cdot 90} = 180$.

11. Solve simultaneously:

$$\begin{aligned} x + 2y + z &= 14 \\ 2x + y + z &= 12 \\ x + y + 2z &= 18 \end{aligned}$$

   (A) $x = 1$, $y = 3$, $z = 7$ **(B)** $x = -2$, $y = 4$, $z = 8$

   (C) $x = -3$, $y = 12$, $z = 0$ **(D)** $x = 6$, $y = -4$, $z = 4$

   (E) $x = 4$, $y = 4$, $z = 5$

   Solution: A. Add the three equations together to get $4x + 4y + 4z = 44$, so $x + y + z = 11$. Subtract this from each of the given equations, one at a time, to get $x = 1$, $y = 3$, and $z = 7$.

12. How many positive integer divisors does $N = 6^3 \cdot 150$ have?

    (A) 32 **(B)** 75 **(C)** 18 **(D)** 24 **(E)** 27

    Solution: B. The prime factorization of N is $2^3 \cdot 3^3 \cdot 2 \cdot 3 \cdot 5^2 = 2^4 3^4 5^2$. Thus N has $(4+1)(4+1)(2+1) = 75$ divisors.

13. Let x_1 and x_2 be the real solutions of the equation $x^2 + bx + c = 0$ with $b \neq 0$. If $x_1 - x_2 = 4$ and $x_1^2 + x_2^2 = 40$, then b must be equal to

    (A) 8 **(B)** 4 **(C)** 12 **(D)** 8 or 4 **(E)** 8 or -8

    Solution: E. By substitution of the givens, we have $x_1^2 + (x_1 - 4)^2 = 40$ and $x_1 = 6, -2$. Then $x_2 = x_1 - 4 = 2, -6$. Since $-b = x_1 + x_2$ and $b \neq 0$, we must have either $-b = 6 + 2 = 8$ or $-b = -2 - 6 = -8$.

14. Suppose A and B are sets with 5 and 7 elements respectively and $A \cap B$ has 2 elements. How many elements does $A \cup B$ have?

    (A) 14 **(B)** 12 **(C)** 8 **(D)** 10 **(E)** 18

    Solution: D. The inclusion-exclusion principle states that for any pair of sets A and B, $|A \cup B| = |A| + |B| - |A \cap B|$, which in this case yields $|A \cup B| = 5 + 7 - 2 = 10$.

15. For what positive value of x is there a right triangle with sides $2x + 2$, $6x$, and $6x + 2$?

    (A) 8 **(B)** 6 **(C)** 4 **(D)** 5 **(E)** 9

    Solution: C. The number x must satisfy $(2x+2)^2 + (6x)^2 = (6x+2)^2$ since $6x + 2$ is clearly the largest of the three numbers. Thus $4x^2 + 8x + 4 + 36x^2 - 36x^2 - 24x - 4 = 4x^2 - 16x = 0$ from which it follows that $x = 4$.

16. Find an equation of the line tangent to the circle $x^2 + y^2 = 2$ at the point $(1, 1)$.

    (A) $3x - y = 4$ **(B)** $y = 2x - 1$ **(C)** $y = -2x + 3$

    (D) $3x - 2y = 1$ **(E)** $x + y = 2$

    Solution: E. The center of our circle is $(0, 0)$ and the line connecting the center of the circle with $(1, 1)$ has slope 1. Thus the tangent line must have slope -1 and pass through $(1, 1)$. The point-slope form of the equation of the tangent line is $y - 1 = -(x - 1)$, which may be rearranged into $x + y = 2$.

17. Three points $A = (0, 1)$, $B = (2, a)$ and $C = (3, 7)$ are on a straight line. What is the value of a?

    (A) 5 **(B)** 3 **(C)** 1 **(D)** 4 **(E)** 2

    Solution: A. The slope of $\overline{AB}$ is the same as that of $\overline{AC}$. That is, $\dfrac{a-1}{2-0} = \dfrac{7-1}{3-0}$, and it follows that $a - 1 = 4$ so $a = 5$. Alternatively, use A and C to get the slope and then use A to get the slope-intercept equation $y = 2x + 1$. Then just plug in $x = 2$. Also, there is a graphical approach: draw the line and notice that the line goes through $(2, 5)$.

18. A rope maker cut a cord into three pieces. Let's name the pieces X, Y, and Z. X is 3 feet long, Y is 3 feet longer than one-fourth of Z. Z is as long as X and Y together. How long is the cord?

    (A) 13 feet **(B)** 14 feet **(C)** 15 feet **(D)** 16 feet **(E)** 17 feet

    Solution: D. $X = 3$, $Y = 3 + (1/4)Z$, and $Z = X + Y = 3 + Y$, so $Z = 3+3+(1/4)Z$, so $Z = 6+(1/4)Z$. So $(3/4)Z = 6$, so $Z = 8$. So $Y = 3+2 = 5$. The length of the cord is 3 + 5 + 8 =16.

19. Find all points (x, y) that have an x-coordinate twice the y-coordinate and that lie on the circle of radius of 5 with center at $(2, 6)$.

    (A) $(6, 3)$ only **(B)** $(2, 1)$ only **(C)** $(4, 2)$ and $(6, 3)$

    (D) $(2, 1)$ and $(0, 0)$ **(E)** $(6, 3)$ and $(2, 1)$

    Solution: E. The two conditions are $(x - 2)^2 + (y - 6)^2 = 25$ and $x = 2y$. Substituting and solving we have: $(2y - 2)^2 + (y - 6)^2 = 25$, $y^2 - 4y + 3 = 0$, $(y - 3)(y - 1) = 0$ so $y = 3$ and $y = 1$. The corresponding x values are 6 and 2.

20. Four numbers are written in a row. The average of the first two numbers is 7, the average of the middle two numbers is 2.3, and the average of the last two numbers is 8.4. What is the average of the first number and the last number?

    (A) 13.1 **(B)** 7.7 **(C)** 8.85 **(D)** 2.3 **(E)** none of the above

    Solution: A. Let A, B, C, D represent the numbers, respectively. Then $(A+B)/2 = 7$, $(B+C)/2 = 2.3$, and $(C+D)/2 = 8.4$.

    $$(A+D)/2 = (A+B)/2 - (B+C)/2 + (C+D)/2 = 7 - 2.3 + 8.4.$$

    So $(A+D)/2 = 13.1$.

21. What non-zero real value for x satisfies $(7x)^{14} = (14x)^7$?

    (A) 1/7 **(B)** 2/7 **(C)** 1 **(D)** 7 **(E)** 14

    Solution: B. Taking the seventh root of both sides, we get $(7x)^2 = 14x$. Simplifying gives $49x^2 = 14x$, which then simplifies to $7x = 2$. Thus $x = 2/7$.

22. The sum of three numbers is 20. The first is four times the sum of the other two. The second is seven times the third. What is the product of all three?

    (A) 28 **(B)** 32 **(C)** 60 **(D)** 84 **(E)** 140

    Solution: A. Let the three numbers be x, y, and z. The given information tells us that

    $$\begin{aligned} y &= 7z \\ x &= 4(y+z) = 4(7z+z) = 32z \end{aligned}$$

    Hence, $x+y+z = 32z+7z+z = 40z = 20$. Hence, $z = 1/2$. Substitution gives $y = 7/2$, $x = 16$. Therefore the product of all three numbers is $(16)(7/2)(1/2) = 28$.

23. Jimmy and Katie run a circular track of circumference 400 meters. Jimmy runs 275 meters per minute, and Katie runs 300 meters per minute. If they started the race at the same start line and run in the same direction, how many minutes would it take for them to meet at the same spot?

    (A) 10 min **(B)** 12 min **(C)** 14 min **(D)** 16 min **(E)** 20 min

    Solution: D. Let the time that they meet be t. Then, $(300 - 275)t = 400$ and $t = 16$. They meet after 16 min.

24. How many integers from 101 to 999 contain the digit 8?

    (A) 270 **(B)** 252 **(C)** 243 **(D)** 219 **(E)** 180

    Solution: B. Consider all three-digit numbers and count the numbers without 8. We have 900 three-digit numbers, and $8 \times 9 \times 9 = 648$ numbers do not contain 8. Thus, $900 - 648 = 252$ numbers have the digit 8.

25. If $\log\left(3\sqrt{3\sqrt{3\sqrt{3\sqrt{3}}}}\right) = A\log 3$, what is A?

    (A) $\frac{31}{16}$ **(B)** $\frac{31}{32}$ **(C)** $\frac{16}{15}$ **(D)** $\frac{32}{17}$ **(E)** $\frac{13}{32}$

    Solution: A. Since $3\sqrt{3\sqrt{3\sqrt{3\sqrt{3}}}} = 3 \cdot 3^{\frac{1}{2}} \cdot 3^{\frac{1}{4}} \cdot 3^{\frac{1}{8}} \cdot 3^{\frac{1}{16}} = 3^{1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}} = 3^{\frac{31}{16}}$, we have $\log 3^{\frac{31}{16}} = \frac{31}{16}\log 3$ and $A = \frac{31}{16}$.