UNC Charlotte 2014 Comprehensive Exam

March 3, 2014

1. The origin and the points where the line ℓ intersects the x-axis and the y-axis are the vertices of a right triangle T whose area is 5. Also the line ℓ is perpendicular to the line given by the equation $5x - y = 15$. What is the length of the hypotenuse of T?

(A) $\sqrt{20}$ (B) $\sqrt{26}$ (C) $\sqrt{29}$ (D) $\sqrt{45}$ (E) $\sqrt{52}$

Solution: E. Let $(f, 0)$ be the x-intercept of ℓ and let $(0, g)$ be the y-intercept of ℓ. Then the area of T is $|fg/2|$ and the length of the hypotenuse is $\sqrt{f^2 + g^2}$. There is an equation for ℓ of the form $x + 5y = C$ for some number C with $f = C = 5g$. So $5 = 5g^2/2$ and from this we get $g = \pm\sqrt{2}$ and $f = \pm 5\sqrt{2}$. Thus the hypotenuse has length $\sqrt{52}$.

2. A two-digit number is written at random. What is the probability that the sum of its digits is 5?

(A) $\frac{5}{89}$ (B) $\frac{1}{18}$ (C) $\frac{6}{89}$ (D) $\frac{1}{15}$ (E) $\frac{4}{89}$

Solution: B. There are 90 two-digit numbers from 10 to 99. Among them there are five whose digits' sum equals 5: $50, 41, 32, 21, 14$. Thus, the probability is $\frac{5}{90} = \frac{1}{18}$.

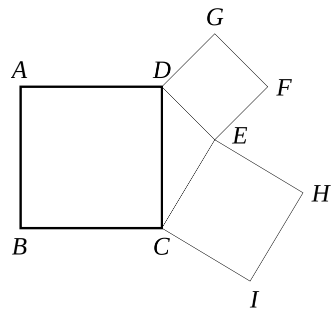
3. Two people toss fair coin. One threw it 10 times while the other – 11 times. What is the probability that the second coin's tail appeared more times than the first?

(A) $\frac{511}{1024}$ (B) $\frac{513}{1024}$ (C) $\frac{1025}{2028}$ (D) $\frac{1027}{2048}$ (E) $\frac{1}{2}$

Solution: E. The events "the second person has more tails than the first" and "the second person has more heads than the first" have the same probability and compliment each other. Hence, the probability of each event is $1/2$.

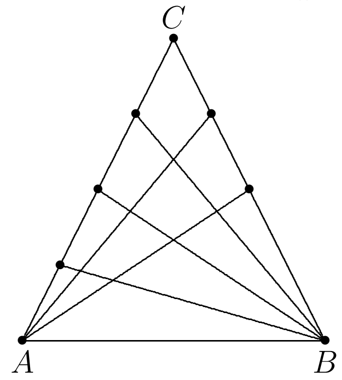
4. Three squares are arranged as in the diagram. The largest ($\square$ ABCD) has area 36 and the smallest ($\square$ DEFG) has area 8. If triangle $\triangle$CDE has area 6, what is the area of the other square ($\square$ CEHI)?

(A) 18 (B) 20 (C) 22 (D) 24 (E) 28



Solution: B. The area of the middle sized square is the square of the length of the segment $\overline{CE}$. Also $\overline{CD}$ has length 6 and $\overline{DE}$ has length $\sqrt{8}$. Let J be the foot of the perpendicular from E to $\overline{CD}$. Also let t be the length of $\overline{EJ}$, z be the length of $\overline{CE}$ and x be the length of $\overline{DJ}$. We have the following relations: $6t = 2 \cdot 6 \quad 8 = x^2 + t^2 \quad z^2 = (6-x)^2 + t^2 = 36 - 12x + x^2 + t^2 = 36 - 12x + 8 = 44 - 12x$ Then $t = 2$, so $x^2 = 8 - 4 = 4$. Finally, $z^2 = 44 - 24 = 20$.

5. The figure below is built with eight line segments, each with either A or B as an end-point. Triangles of various sizes are formed with parts of these segments as boundary.



How many triangular regions are there?

(A) 19 (B) 32 (C) 36 (D) 38 (E) 42

Solution: E. Each triangle contain A or B or both. There are $6 \cdot 4 = 24$ that contain A and $10 \cdot 3 = 30$ that contain B. But we are double counting $4 \cdot 3 = 12$ triangles so there are $24 + 30 - 12 = 42$ triangular regions.

6. Cucumbers contain 99% of water. After being exposed to the sun the amount of water drops to 98%. What percentage of weight did the cucumbers lose?

(A) 0.98% (B) 1% (C) 2% (D) 4% (E) 50%

Solution: E. The non-watery part of fresh cucumbers makes up 1% of their weight. In "dry" cucumbers *the same* non-watery amount represents 2% of the weight. That means the weight halves.

7. Suppose a is a real number for which $a^2 + \frac{1}{a^2} = 14$. What is the largest possible value of $a^3 + \frac{1}{a^3}$?

(A) 48 (B) 52 (C) 56 (D) 60 (E) 64

Solution: B. Compute $(a + 1/a)^2$ to get $a^2 + 2 + 1/a^2 = 14 + 2$, so $a + 1/a$ is either 4 or -4. Then $(a+1/a)^3 = a^3 + 3a + 3/a + 1/a^3 = a^3 + 1/a^3 + 12 = 4^3 = 64$, so $a^3 + \frac{1}{a^3} = 52$.

8. Let $p(x) = (x-7)(x^3 + 5x^2 + 7x - 11) + (x-9)(x^3 + 5x^2 + 7x - 11)$. What is the sum of the roots of $p(x) = 0$?

(A) -11 (B) -3 (C) 3 (D) 11 (E) 13

Solution: C. Factor $p(x)$ to get $p(x) = (2x - 16)(x^3 + 5x^2 + 7x - 11)$, which has sum of roots $8 - 5 = 3$.

9. Suppose a, b, c are integers satisfying $1 \leqslant a < b < c$ and $a^2 + b^2 + c^2 = 14(a + b + c)$. What is the sum $a + b + c$?

(A) 38 (B) 39 (C) 40 (D) 41 (E) 42

Solution: A. Rewrite the equation as $(a-7)^2+(b-7)^2+(c-7)^2 = 147$ after completing the square three times. So we're looking for three squares whose sum is 147. Note that all three are odd or just one is odd. Trying all three odd leads to $11^2 + 5^2 + 1^2 = 121 + 25 + 1 = 147$. This is the only solution.

10. What is the largest prime divisor of $2^{16} - 16$?

(A) 7 (B) 13 (C) 17 (D) 19 (E) 23

Solution: B. $2^{16} - 16 = 2^4(2^{12} - 1) = 2^4(2^6 - 1)(2^6 + 1) = 2^4 \cdot 63 \cdot 65 = 2^4 \cdot 9 \cdot 7 \cdot 5 \cdot 13.$

11. Suppose the roots of the equation $(x^2 - 2x + m)(x^2 - 2x + n) = 0$, where m, n are two real numbers, form an arithmetic sequence with the first term being $\frac{1}{4}$. Then $|m - n| =$

(A) 1 (B) $\frac{3}{4}$ (C) $\frac{1}{2}$ (D) $\frac{3}{8}$ (E) 2

Solution: C. Suppose the 4 roots are given by $\frac{1}{4}, \frac{1}{4} + d, \frac{1}{4} + 2d, \frac{1}{4} + 3d,$ where d is a real number. Further suppose the roots of $x^2 - 2x + m = 0$ are r_1, r_2 and the roots of $x^2 - 2x + n = 0$ are r_3, r_4, then from $r_1 + r_2 + r_3 + r_4 = 2 + 2$, we have $d = \frac{1}{2}$. Since $r_1 + r_2 = 2$, $r_3 + r_4 = 2$, and $r_1, r_2, r_3, r_4 \in \left\{\frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \frac{7}{4}\right\}$, there are only two possibilities: either $r_1, r_2 \in \{\frac{3}{4}, \frac{5}{4}\}$, $r_3, r_4 \in \{\frac{1}{4}, \frac{7}{4}\}$; or $r_1, r_2 \in \{\frac{1}{4}, \frac{7}{4}\}$, $r_3, r_4 \in \{\frac{3}{4}, \frac{5}{4}\}$. In both two cases, we have $|m - n| = |r_1 \cdot r_2 - r_3 \cdot r_4| = \frac{1}{2}$.

12. If $\sin x + \cos x = 1.2$, then what is the value of $\sin 2x$?

(A) 0.22 (B) 0.88 (C) -0.2 (D) 0.44 (E) -0.88

Solution: D. $1.2^2 = (\sin x + \cos x)^2 = 1 + 2\sin x \cos x = 1 + \sin 2x$. Hence, $\sin 2x = 1.2^2 - 1 = 0.44$.

13. Let $i = \sqrt{-1}$. Then $(\sin 6^\circ + i\cos 6^\circ)^{30} =$

(A) 1 (B) i (C) -1 (D) $\frac{\sqrt{2}}{2}$ (E) $-\frac{\sqrt{2}}{2}$

Solution: A. $(\sin 6^\circ + i\cos 6^\circ)^{30} = [i \cdot (\cos 6^\circ - i\sin 6^\circ)]^{30} = i^{30}(e^{-i6^\circ})^{30} = -e^{-i\pi} = 1.$

14. The sides of a right triangle are a, $2a + 2d$ and $2a + 3d$, with a and d both positive. The ratio of a to d is:

(A) $5:1$ (B) $27:2$ (C) $4:1$ (D) $1:5$ (E) $2:3$

Solution: A. For the right triangle, the longest side is the hypotenuse, therefore $a^2 + (2a + 2d)^2 = (2a + 3d)^2,\ a^2 + (4a^2 + 8ad + 4d^2) = 4a^2 + 12ad + 9d^2,\ a^2 - 4ad - 5d^2 = 0,\ (a - 5d)(a + d) = 0,\ a = 5d$ or $a = -d$ (dropped since both a and d >0). $a : d = 5d : d = 5 : 1$.

15. Two armies are advancing towards each other, each one at 1 mph. A messenger leaves the first army when the two armies are 10 miles apart and runs towards the second at 9 mph. Upon reaching the second army, he immediately turns around and runs towards the first army at 9 mph. How many miles apart are the two armies when the messenger gets back to first army?

(A) 5.6 (B) 5.8 (C) 6 (D) 6.2 (E) 6.4

Solution: E. The messenger reaches the second army in one hour (the messenger and the second army advance at each other at combined speed of 10 mph). At that time, the two armies are 8 miles apart. It takes 0.8 hours for the messenger to get back to the first army. At that time the armies are $8 - 0.8 - 0.8 = 6.4$ miles apart.

16. A machine was programmed to transmit a certain sequence of five digits, all zeros and ones, five times. One time it did it correctly; one time it did so with one mistake; one time it did so with two mistakes; one time it did so with three mistakes; one time it did so with four mistakes. The five transmissions are listed below. Which is the correct sequence?

(A) 00001 (B) 00100 (C) 01100 (D) 10010 (E) 10011

Solution: B. Let us call ”distance“ between transmissions the number of positions in which they differ. We need to find which of the five transmissions has distances to the other four transmissions, 1, 2, 3, and 4 in some order. Now, the distances from (a) to the other transmissions are 2,3,2,3; from (b): 2,1,3,4; from (c): 3,1,4,5; from (d): 3,3,4,1; from (e): 2,4,5,1. Thus, the answer is B.

17. Oil is pumped into a non-empty tank at a changing rate. The volume of oil in the tank doubles every minute and the tank is filled in 10 minutes. How many minutes did it take for the tank to be half full?

(A) 2 (B) 5 (C) 7 (D) 8 (E) 9

Solution: E. At the 10 minute mark, the tank has twice as much oil in it as it did at the 9 minute mark. So the tank became half-full at the 9 minute mark.

18. Jack and Lee walk around a circular track. It takes Jack and Lee respectively 6 and 10 minutes to finish each lap. They start at the same time, at the same point on the track, and walk in the same direction around the track. After how many minutes will they be at the same spot again (not necessarily at the starting point) for the first time after they start walking?

(A) 15 (B) 16 (C) 30 (D) 32 (E) 60

Solution: A. Experimenting with the numbers in turn, dividing 6 into 15 and 10 into 15, we get answers one whole number apart, so they are together again at 15 minutes.

19. Imagine that Rubik’s cube consists of 27 equal cubes. Find the minimal amount of hits of an axe that is needed in order to divide Rubik’s cube into 27 equal cubes?

(A) 6 (B) 8 (C) 10 (D) 12 (E) 16

Solution: A. The small cube in the middle has 6 faces. On the other hand it is easy to see how to strike 6 times to divide the cube into 27 pieces.

20. A triangular pyramid is placed inside a sphere so that it does not intersect or touches the sphere. Each of the faces of the pyramid is extended so that they become planes. In how many pieces will the extensions cut the sphere?

(A) 9 (B) 10 (C) 12 (D) 14 (E) 15

Solution: D. A piece of the sphere is situated next to the each vertex, each face and each edge of the pyramid.

21. Let $a_n = \dfrac{1}{\sqrt{n}+\sqrt{n+1}}$. Find the sum $a_1 + a_2 + \cdots + a_{99}$.

(A) 6 (B) 8 (C) 9 (D) 12 (E) 15

Solution: C. It is easy to see that $a_n = \sqrt{n+1} - \sqrt{n}$. Consequently, $\displaystyle\sum_1^{99} a_n = \sqrt{100} - 1$.

22. Evaluate exactly $\sqrt{5+2\sqrt{6}} + \sqrt{5-2\sqrt{6}}$.

(A) $2\sqrt{2}$ (B) $3\sqrt{2}$ (C) $2\sqrt{3}$ (D) $2\sqrt{5}$ (E) $\frac{3}{2}\sqrt{6}$

Solution: C. Denoting $x = \sqrt{5+2\sqrt{6}} + \sqrt{5-2\sqrt{6}}$, we have $x^2 = 5 + 2\sqrt{6} + 5 - 2\sqrt{6} + 2\sqrt{5+2\sqrt{6}}\sqrt{5-2\sqrt{6}} = 10 + 2\sqrt{5^2+4\cdot 6} = 12$. Hence, $x = \sqrt{12} = 2\sqrt{3}$.

23. What is the largest power of 2 that divides the number $K = 75! - 71!$?

(A) 2^{62} (B) 2^{63} (C) 2^{65} (D) 2^{66} (E) 2^{67}

Solution: E. We can write $K = 75\cdot 74\cdot 73\cdot 72\cdot 71! - 71! = 71!(T-1)$, where T is the even number $75\cdot 74\cdot 73\cdot 72$. So the answer we seek is the largest power of 2 that divides 71!. This is the number $\lfloor\frac{71}{2}\rfloor + \lfloor\frac{71}{4}\rfloor + \lfloor\frac{71}{8}\rfloor + \lfloor\frac{71}{16}\rfloor + \lfloor\frac{71}{32}\rfloor + \lfloor\frac{71}{64}\rfloor = 35+17+8+4+2+1 = 67$.

24. Consider the sequence $a_1 = 1, a_2 = 13, \ldots$ where each term a_n is obtained from the previous term a_{n-1} by appending the n^{th} odd number. So $a_3 = 135, a_4 = 1357$, etc. Find the number m so that a_m is the 30^{th} multiple of 9 in the sequence.

(A) 66 (B) 77 (C) 81 (D) 90 (E) 99

Solution: D. Note that $a_n \equiv \sum_{k=1}^{n}(2k-1) \equiv n^2$ (mod 9), and the thirtieth multiple of 9 among these is 90.

25. Let P denote the point on the circle $x^2 + 2x + y^2 - 4y = 20$ that is closest to $(7,8)$. What is the slope of the line passing through both P and $(7,8)$?

(A) 2/3 (B) 3/4 (C) 4/5 (D) 1 (E) 2

Solution: B. Completing the squares, the circle can be written $(x+1)^2 + (y-2)^2 = 25$. The line through $(7, 8)$ and the center of the circle $(-1, 2)$ includes the point P, so the slope is $\frac{8-2}{7+1} = \frac{3}{4}$.