UNC Charlotte 2008 Comprehensive

March 3, 2008

1. Suppose a and b are digits satisfying $1 < a < b < 8$. Also, the sum $1111 + 111a + 111b + \cdots$ of the smallest eight four-digit numbers that use only the digits $\{1, a, b, 8\}$ is 8994. What is $a + b$?

   (A) 6 **(B)** 7 **(C)** 8 **(D)** 9 **(E)** 10

   Solution: C. The eight smallest numbers are $1111, 111a, 111b, 1118, 11a1, 11aa, 11ab$ and $11a8$. Their sum is $8858 + 42a + 2b = 8994$, so $42a + 2b = 136$ from which it follows that $21a + b = 68$. From this we can reason that a must be 3, and that b must be 5, so $a + b = 8$.

2. On a die, 1 and 6, 2 and 5, 3 and 4 appear on opposite faces. When 2 dice are thrown, multiply the numbers appearing on the top and bottom faces of the dice as follows:

   (a) number on top face of 1st die $\times$ number on top face of 2nd die
   (b) number on top face of 1st die $\times$ number on bottom face of 2nd die
   (c) number on bottom face of 1st die $\times$ number on top face of 2nd die
   (d) number on bottom face of 1st die number $\times$ on bottom face of 2nd die.

   What can be said about the sum S of these 4 products?

   (A) The value of S depends on luck and its expected value is 48

   (B) The value of S depends on luck and its expected value is 49

   (C) The value of S depends on luck and its expected value is 50

   (D) The value of S is 49

   (E) The value of S is 50

   Solution: D. Suppose U and B are the up and bottom on the first die and u and b for the second. Then the sum S equals $Uu + Ub + Bu + Bb = (U + B)(u + b) = 7 \cdot 7 = 49$.

3. Let N be the largest 7-digit number that can be constructed using each of the digits $1, 2, 3, 4, 5, 6$, and 7 such that the sum of each two consecutive digits is a prime number. What is the reminder when N is divided by 7?

   (A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

   Solution: E. The number is $N = 7652341$, which is constructed from left to right. The reminder is 4.

4. For how many n in $\{1, 2, 3, \ldots, 100\}$ is the tens digit of n^2 odd?

   (A) 16 **(B)** 17 **(C)** 18 **(D)** 19 **(E)** 20

   Solution: E. There are 2 in each decile, $10a + 4$ and $10a + 6$. The tens digit of $(10a + 4)^2 = 100a^2 + 80a + 16$ is the units digit of $8a + 1$, while the tens digit of $(10a + 6)^2 = 100a^2 + 120a + 36$ is the units digit of $2a + 3$, both of which are odd for any integer a. All the other tens digits of perfect squares are even: $(10a + b)^2 = 100a^2 + 20a + b^2$, the tens digit of which is the tens digit of $2a + b^2$, which is even if the tens digit of b^2 is even. But the tens digit of b^2 is even if $b \neq 4, b \neq 6$.

5. How many pairs of positive integers (a, b) with $a + b \leq 100$ satisfy

$$\frac{a + b^{-1}}{a^{-1} + b} = 13?$$

(A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 7

Solution: E. Multiplying the given equality $a + b^{-1} = 13(b + a^{-1})$ by ab we obtain: $a(ab+1) = 13b(ab+1)$, or $(a - 13b)(ab+1) = 0$. Since $ab+1 > 0$, the given equation is equivalent to $a = 13b$. The inequality $a+b \leq 100$ means that $14b \leq 100$; therefore, the possible values of the positive integer b are $1, 2, \ldots, 7$, and there are 7 solutions: $(13, 1)$, $(26, 2), \ldots, (91, 7)$.

6. The numbers $1, 2, 4, 8, 16, 32$ are arranged in a multiplication table, with three along the top and the other three down the column. The multiplication table is completed and the sum of the nine entries is tabulated. What is the largest possible sum obtainable.

(A) 902 **(B)** 940 **(C)** 950 **(D)** 980 **(E)** 986

$\times$	a	b	c
d			
e			
f			

Solution: D. The sum is $(a + b + c) \cdot (d + e + f)$ which is as large as possible when the two factors $a + b + c$ and $c + d + e$ are as close together as possible (given that the sum is constant(63)). We must pair the 1 and the 2 with 32 to get $35 \cdot 28 = 980$.

7. An unlimited supply of struts of lengths $3, 4, 5, 6$, and 7 are available from which to build (nondegenerate) triangles. How many noncongruent triangles can be built?

   (A) 21 **(B)** 25 **(C)** 28 **(D)** 30 **(E)** 32

   Solution: If we think of each triangle we can build as a three digit number where the digits are nondecreasing as we move from left to right. They are $333, 334, 335, 344, 345, 346, \ldots$. There are 35 of these, and they fall into three types: 10 of the type (a, b, c) with $a < b < c$, 20 of the type (a, a, b) or (a, b, b) with $a < b$ and 5 of the type (a, a, a). Every such number built from the digits $3, 4, 5, 6$ and 7 corresponds to a triangle in this way except $336, 337$, and 347, because these three fail to satisfy the triangle inequality $a + b > c$. There are $35 - 3 = 32$ of these numbers.

8. Five points lie on a line. When the 10 distances between each pair of them are computed and listed from smallest to largest we obtain

   $$2, 4, 5, 7, 8, k, 13, 15, 17, 19.$$

   What is k?

   (A) 9 **(B)** 10 **(C)** 11 **(D)** 12 **(E)** 13

   Solution: Without loss of generality, let the left-most coordinate is 0. Since 19 is the largest difference, we can assume that 19 is in the 5-element set. Let $S = \{0, a, b, c, 19\}$ denote the set, with $a < b < c$. Then the set of differences of members of S is $\{a, b, c, 19-a, 19-b, 19-c, 19, c-a, c-b, b-a\}$ and the sum of all these is $4 \cdot 19 + 2c - 2a = 76 + 2(c-a)$. On the other hand, the sum of the given distances is $2+4+5+\cdots = 90+k$. Therefore $76+2(c-a) = 90+k$, and it follows that $2(c-a) = 14+k$. This implies that k is even. Since $9 \leq k \leq 12$, we know that either $k = 10$ or $k = 12$. If $k = 10$, then $2(c - a) = 24$ and $c - a = 12$ which contradicts the listing of differences given. Therefore, $k = 12$ is the only possibility. Some trial and error leads to the two possible coordinate sets $\{0, 2, 7, 15, 19\}$ and $\{0, 4, 12, 17, 19\}$. The missing distance is $k = 12$.

9. Once a strange notebook was found. It contained exactly the following 100 statements:
"This notebook has exactly one false statement."
"This notebook has exactly two false statements."
"This notebook has exactly three false statements."
⋮
"This notebook has exactly 100 false statements." How many true statements are there in the notebook?

   (A) 0 **(B)** 1 **(C)** 50 **(D)** 99 **(E)** 100

   Solution: B. Any two statements in the notebook contradict to each other. That means that there can be not more than one true statements. The remaining case – no true statements – is impossible: if this were true, the statement "This notebook contains 100 false statements" would be true, but this statement *is* in the notebook, so it would be false – a contradiction.Since there is one true statement, there are 99 are false statements. Such a statement exists in the notebook: "This notebook has exactly 99 false statements."

10. Color the surfaces of a cube of dimension $5 \times 5 \times 5$ red, and then cut the cube into unit cubes. Remove all the unit cubes with no red faces. Use the remaining cubes to build a cuboid (a rectangular brick), keeping the outer surface of the cuboid red. What is the maximum possible volume of the cuboid?

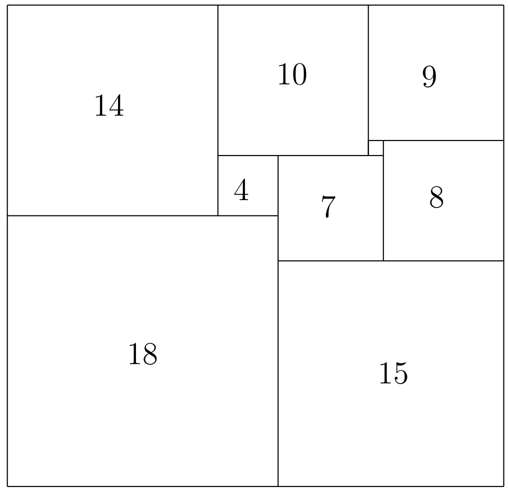
    (A) 70 **(B)** 80 **(C)** 92 **(D)** 96 **(E)** 98

    Solution: D. We have $5^3 - 3^3 = 98$ unit cubes with some red faces. Among these there are 8 corner cubes with three painted faces, and of course, these must be used as the corners of the brick. There are $12 \cdot 3 = 36$ unit cubes with two adjacent red faces, and there are $6 \cdot 9 = 54$ unit cubes with one red face. The maximum volume is therefore no more than $8 + 36 + 54 = 98$. The only brick we could hope to build with volume 98 is a $7 \times 7 \times 2$, but this would require 40 unit cubes with two adjacent red faces. Of course a brick with volume 97 is hopeless. How about 96? Could it be a $6 \times 4 \times 4$ brick? How many 'edge cubes' are needed? Answer: $2 \cdot 2 \cdot 4 + 4 \cdot 4 = 32$. Thus we have 4 two-faced cubes left that we can use with the 54 one red face cubes. This is enough to fill out the rest of the brick.

11. Construct a rectangle by putting together nine squares with sides equal to 1, 4, 7, 8, 9, 10, 14, 15 and 18. What is the sum of the areas of the squares on the 4 corners of the resulting rectangle.

    (A) 626 **(B)** 746 **(C)** 778 **(D)** 810 **(E)** 826

Solution: E. Factor the sum of the squares to get $1^2 + 4^2 + 7^2 + \cdots + 18^2 = 1056 = 2^5 \cdot 3 \cdot 11$. Notice that the only pair of dimensions that will accommodate the 18×18 square together with both the 14×14 and the 15×15 squares is 32×33. The four corners are unique. The only way to make room for the three largest squares is to put them in corners with the 14×14 square and the 15×15 square next to the 18×18 square. See the figure below. The only two squares that could fill the 3×15 gap left above the 15×15 square are the 7×7 and the 8×8 squares. Then the 1×1 must go in the tiny hole left. Finally the 10×10 and the 9×9 squares can be placed. So the sum of the areas of the four corner squares is 826.



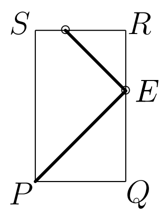
12. A number N is divisible by 90, 98 and 882 but it is NOT divisible by 50, 270, 686 and 1764. It is known that N is a factor of 9261000. What is N?

(A) 4410 **(B)** 8820 **(C)** 22050 **(D)** 44100 **(E)** 88200

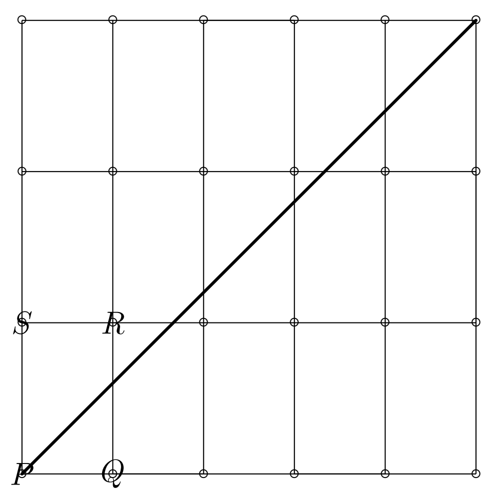
Solution: A. Factor each of the numbers into primes: $50 = 2 \cdot 5^2; 90 = 2 \cdot 3^2 \cdot 5; 98 = 2 \cdot 7^2; 270 = 2 \cdot 3^3 \cdot 5;$ $686 = 2 \cdot 7^3;$ $882 = 2 \cdot 3^2 \cdot 7^2$ and $1764 = 2 \cdot 3^2 \cdot 7^2$. So $N = k \cdot 2 \cdot 3^2 \cdot 5 \cdot 7^2$ and the non-divisibility conditions imply that none of $2, 3, 5$, and 7 are divisors of k. Since $9261000 = 2^3 \cdot 3^3 \cdot 5^3 \cdot 7^3$, it follows that only $k = 1$ works and $N = 4410$.

13. On a rectangular table $PQRS$ that is 5 units long and 3 units wide, a ball is rolled from point P at an angle of 45° toward the point E, and bounces off SR at an angle of 45° as shown below. The ball continues to bounce off the sides at 45° until it reaches a corner. How many times does the ball bounce?

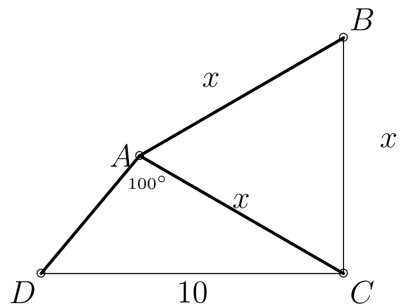
    (A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 9



Solution: C. Think of the 3×5 box as part of a grid of such boxes in the plane. The line from $(0, 0)$ to $(15, 15)$ hits the boundaries of these boxes 6 times, and each one of these corresponds to a bounce. You can see that the ball will hit the pocket R after six bounces.



14. Triangle ABC below is equilateral and the length of each side is x. Angle BCD is a right angle and angle DAC is 100 degrees. The side DC has length 10. Find x. Round your answer to 2 decimal places.



(A) 7.16 **(B)** 7.32 **(C)** 7.51 **(D)** 7.78 **(E)** 7.95

Solution: D. Because triangle ABC is equilateral, angle ACB is 60 degrees. Because angle BCD is 90 degrees, angle ACD is 30 degrees. Therefore angle ADC is 50 degrees. Using the law of sines, $\frac{x}{sin(50)} = \frac{10}{sin(100)}$. Consequently, $x \approx 7.78$.

15. Joe lives near a river where he goes swimming every day: he swims 1 mile upstream, 1 mile downstream, and exits the river the same place as he entered it. Recently Joe went on a vacation to a lake, where he noticed that during his workouts lasting the same time, swimming at the same constant speed, he is able to swim 2.2 miles each day. How much faster is Joe swimming than the speed of the river? Round your answer to two decimal places.

    (A) 2.5 times **(B)** 3.32 times **(C)** 3.5 times

    (D) 4.1 real times **(E)** 11 times

    Solution: B. Denote Joe's speed by v_1, the river's speed by v_2. Swimming 1 mile upstream and 1 mile downstream takes $1/(v_1 - v_2) + 1/(v_1 + v_2)$ units of time, swimming 2.2 miles in a lake takes $2.2/v_1$ units of time. The equation

    $$\frac{1}{v_1 - v_2} + \frac{1}{v_1 + v_2} = \frac{2.2}{v_1}$$

    can be re-written as

    $$\frac{1}{1 - v_2/v_1} + \frac{1}{1 + v_2/v_1} = 2.2,$$

    which implies that $(v_2/v_1)^2 = 1/11$ and thus $v_1/v_2 = \sqrt{11} \approx 3.32$.

16. A particle P moves from the point $A = (0, 4)$ to the point $B = (10, -4)$. The particle P can travel the upper half place $\{(x, y) \mid y \geq 0\}$ at the speed of 1 and travel the lower half plane $\{(x, y) \mid y \leq 0\}$ at the speed of 2. Find the point $C = (c, 0)$ on the x-axis which would minimize the sum of squares of the travel times of the upper and lower half plane.

    (A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

    Solution: B. The sum of squares of travel times is

    $$(\frac{\sqrt{c^2+16}}{1})^2 + (\frac{\sqrt{(10-c)^2+16}}{2})^2 = \frac{5}{4}c^2 - 5c + 45 = \frac{5}{4}(c^2 - 4c + 36)$$

    Therefore the sum of squares of travel times is minimized when $c = 2$.

17. During recess, one of five pupils wrote something nasty on the chalkboard. When questioned by the class teacher, the following ensured:

A : It was 'B' or 'C'.

B : Neither 'E' nor I did it.

C : You are both lying.

D : No, either A or B is telling the truth.

E : No, 'D', that is not true.

The class teacher knows that three of them never lie while the other two cannot be trusted. Who was the culprit?

(A) A **(B)** B **(C)** C **(D)** D **(E)** E

Solution: C. If D's statement is false, then both A and B are also lying, which would mean that we have three liars, and that is impossible. So D's statement is true and therefore E's statement is false. Since C's statement is also false, it must be that A, B and D are honest. But A says it was B or C and B denies it, so only C is left.

18. Find the maximum possible value of $(xv - yu)^2$ over the surface

$$x^2 + y^2 = 4,\ u^2 + v^2 = 9$$

(A) 26 **(B)** 30 **(C)** 35 **(D)** 36 **(E)** 40

Solution: D.

$$\begin{aligned}(xv - yu)^2 &= x^2v^2 - 2xvyu + y^2u^2 \\ &= (x^2 + y^2)(u^2 + v^2) - x^2u^2 - 2xyuv - y^2v^2 \\ &= 36 - (xu + yv)^2 \\ &\le 36\end{aligned}$$

The maximum occurs when $(u, v) = \frac{3}{2}(-y, x)$.

Alternatively, normalize the variables by putting $x = 2a,\ y = 2b,\ u = 3c,\ v = 3d$, so that $a^2+b^2 = c^2+d^2 = 1$. The problem becomes: to maximize $(ad-bc)^2$, or equivalently, to maximize $|ad - bc|$. Note that

$$|ad| \le \frac{a^2 + d^2}{2}, \quad |bc| \le \frac{b^2 + c^2}{2},$$

so that

$$|ad - bc| \le |ad| + |bc| \le \frac{a^2 + d^2 + b^2 + c^2}{2} = 1.$$

The value 1 is attained, in particular, when $a = d$ and $b = -c$ (say, $a = d = 1,\ b = c = 0$). Therefore, $\max(ad-bc)^2 = 1$, and $\max(xv-yu)^2 = (2\cdot3)^2 = 36$.

We can use trig for yet one more solution. We have $x = 2\cos\alpha,\ y = 2\sin\alpha,\ u = 3\cos\beta,\ v = 3\sin\beta$ with arbitrary α and β, so that

$$xv - yu = 6(\sin\beta\cdot\cos\alpha - \cos\beta\cdot\sin\alpha) = 6\sin(\beta - \alpha),$$

and thus $\max(xv - yu)^2 = 36$.