The Inclusion Exclusion Principle

1. Among 18 students in a room, 7 study mathematics, 10 study science, and 10 study computer programming. Also, 3 study mathematics and science, 4 study mathematics and computer programming, and 5 study science and computer programming. We know that 1 student studies all three subjects. How many of these students study none of the three subjects?

   Solution:
   Solution. Let M, S, and C denote the sets of students who study math, science and computing respectively and let U be the entire set of 18 students. Then $|M| = 7, |S| = 10$, and $|C| = 10$. Also, we have $|MS| = 3, |MC| = 4$, and $|SC| = 5$, where, $|x|$ denotes the number of elements of the set x and juxtoposition of sets means intersection. Finally, $|MCS| = 1$. Then

   $$|U|-(|M|+|S|+|C|-|MS|-|MC|-|SC|+|MSC|) = \overline{MSC}| = 18-(27-12+1) = 2.$$

2. Let A, B, and C be sets with the following properties:

   - $|A| = 100$, $|B| = 50$, and $|C| = 48$
   - The number of elements that belong to exactly one of the three sets is twice the number that belong to exactly two of the sets.
   - The number of elements that belong to exactly one of the three sets is three times the number that belong to all of the sets.

   How many elements belong to all three sets?

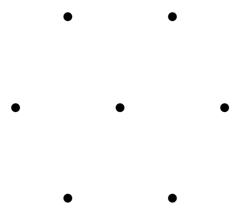
3. Three sets A, B, and C have the following properties: $N(A) = 63, N(B) = 91, N(C) = 44, N(A \bigcap B) = 25, N(A \bigcap C) = 23, N(C \bigcap B) = 21$. Also, $N(A \bigcup B \bigcup C) = 139$. What is $N(A \bigcap B \bigcap C)$?

   Solution: Let x denote the cardinality of $N(A \bigcap B \bigcap C)$. Then x satisfies $x + 15 + 25 + 45 + 21 + 23 = 139$ using a Venn diagram. Thus $x = 10$.

4. Two circles and a triangle are given in the plane. What is the largest number of points that can belong to at least two of the three figures?

   Solution:
   Solution. The number is maximized when no point belongs to all three sets. Since $|TC_1| \leq 6, |TC_2| \leq 6$, and $|C_1C_2| \leq 2$, the maximum possible number is $6 + 6 + 2 = 14$. It is not hard to draw a picture to show that 14 such points are possible.

5. How many equilateral triangles have at least two vertices in the hexagonal lattice shown?



Solution: Solution. There are 6 "area 1" triangles that have the center point as a vertex and 6 more area 1 triangles that don't. There are 4 triangle with area 3, two of whose vertices are in the set $\{B, D, F\}$ and two others with vertices in the set $\{A, C, E\}$. Finally there are 6 triangles of area 4 (two each with edges AD, BE, and CF. Thus the total is $12 + 8 + 6 = 26$.

6. How many integers in the set $\{1, 2, 3, 4, \ldots, 360\}$ have at least one prime divisor in common with 360?

Solution:
Solution. $360 = 2^3 \cdot 3^2 \cdot 5$. Let A_2, A_3, and A_5 denote the multiples of $2, 3$, and 5 respectively among $1, 2, 3, \ldots 360$. To find $|A_2 \cup A_3 \cup A_5|$, apply the principal of inclusion-exclusion for three sets

$$|A_2 \cup A_3 \cup A_5| = |A_2| + |A_3| + |A_5| - |A_2A_3| - A_2A_5| - |A_3A_5| + |A_2A_3A_5|$$

. Note that the number of multiples of n less than or equal to N is $\lfloor \frac{N}{n} \rfloor$. Thus $|A_2 \cup A_3 \cup A_5| = \lfloor \frac{360}{2} \rfloor + \lfloor \frac{360}{3} \rfloor + \lfloor \frac{360}{5} \rfloor - \lfloor \frac{360}{6} \rfloor - \lfloor \frac{360}{10} \rfloor - \lfloor \frac{360}{15} \rfloor + \lfloor \frac{360}{30} \rfloor$
$= 180 + 120 + 72 - 60 - 36 - 24 + 12 = 264$.

7. Let $U = \{1, 2, 3, \ldots, 1000\}$ and let A_2, A_3, and A_5 denote the subsets of U defined as follows:
$A_2 = \{n \,|\, 1 \leq n \leq 1000 \text{ and } n \text{ is even } \}$,
$A_3 = \{n \,|\, 1 \leq n \leq 1000 \text{ and } n \text{ is a multiple of } 3\}$,
$A_5 = \{n \,|\, 1 \leq n \leq 1000 \text{ and } n \text{ is a multiple of } 5\}$,
All complements are taken with respect to U. Find the number of elements of each of the sets listed below. a. $A_2 \bigcap A_3 \bigcap A_5$; b. $A_2 \bigcap A_3 \bigcap \overline{A_5}$; c. $A_2 \bigcap \overline{A_3} \bigcap A_5$;
d. $\overline{A_2} \bigcap A_3 \bigcap A_5$; e. $A_2 \bigcap \overline{A_3} \bigcap \overline{A_5}$; f. $\overline{A_2} \bigcap A_3 \bigcap \overline{A_5}$; g. $\overline{A_2} \bigcap \overline{A_3} \bigcap A_5$; and h. $\overline{A_2} \bigcap \overline{A_3} \bigcap \overline{A_5}$.

8. In a math contest, three problems, A, B, and C were posed. Among the participants there were 25 who solved at least one problem. Of all the participants who did not solve problem A, the number who solved problem B was twice

the number who solved C. The number who solved only problem A was one more than the number who solved A and at least one other problem. Of all participants who solved just one problem, half did not solve problem A. How many solved just problem B?

9. How many numbers can be obtained as the product of two or more of the numbers $3, 4, 4, 5, 5, 6, 7, 7, 7$?

   Solution: Solution. Take G as the multiset $\{3, 4, 4, 5, 5, 6, 7, 7, 7\}$, and P as the process P_1 with the modification that we must use at least two members of G and we multiply instead of add. Note that each member n of R uniquely determines the subset S_n of G whose product it is. We claim that each product in R uniquely determines its factors among the multiset. Factor the product n of members of G into primes to get something of the form $n = 2^i 3^j 5^k 7^l$. The exponent i is odd if and only if the 6 appears in the product. The number of 5's and 7's in S_n is just k and l respectively and the number of 4's is $\lfloor \frac{i}{2} \rfloor$. The number of 6's is $i - 2\lfloor \frac{i}{2} \rfloor$, and the number of 3's is j minus the number of 6's. Thus the number of members of R is the number alternative ways to treat the various values. We can include the 3 or not, include the 6 or not, include 0, 1, or 2 of the 4's, 0,1, or 2 of the 5's, and 0,1,2, or 3 of the 7's. This number is

   $$2 \cdot 2 \cdot 3 \cdot 3 \cdot 4 - 1 - 5 = 138.$$

10. (1994 UNC Charlotte Comprehensive Exam) How many of the first 100 positive integers are expressible as a sum of three or fewer members of the set $\{3^0, 3^1, 3^2, 3^3, 3^4\}$ if we are allowed to use the same power more than once. For example, 5 can be represented, but 8 cannot.

    Solution:

    Solution. The number of powers of 3 used is just the sum of the ternary digits. It is useful therefore to consider the numbers from 1 to 26, 27 to 53, 54 to 80, and 81 to 100. Numbers in the range 1 to 26 have ternary representation of the form $(a_2a_1a_0)_3$. How many of these satisfy $a_2 + a_1 + a_0 \leq 3$? There are 16 such numbers. Those in the range 27 to 53 all have the form $(1a_2a_1a_0)_3$. There are 10 for which $a_2 + a_1 + a_0 \leq 2$. The number in the range 54 to 80 have the form $(2a_2a_1a_0)_3$. Only 4 of these satisfy $a_2 + a_1 + a_0 \leq 1$. The numbers from 81 to 100 all have the form $(1a_3a_2a_1a_0)_3$. We want to know how many of that form are less than 100 and satisfy $1 + a_3 + a_2 + a_1 + a_0 \leq 3$. There are 10 numbers in this range which satisfy the conditions. Hence there are $16 + 10 + 4 + 10 = 40$ such numbers altogether.

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11. How many integers can be expressed as a sum of two or more different members of the set $\{0, 1, 2, 4, 8, 16, 31\}$?

12. In a survey of the chewing gum tastes of a group of baseball players, it was found that: 22 liked juicy fruit
25 liked spearmint
39 like bubble gum
9 like both spearmint and juicy fruit
17 liked juicy fruit and bubble gum
20 liked spearmint and bubble gum
6 liked all three
4 liked none of these
How many baseball players were surveyed?

13. Of 28 students taking at least one subject, the number taking Math and English but not History equals the number taking Math but not History or English. No student takes English only or History only, and six students take Math and History but not English. The number taking English and History but not Math is 5 times the number taking all three subjects. If the number taking all three subjects is even and non-zero, how many are taking English and Math but not History?

14. Mr. Brown raises chickens. Each can be described as thin or fat, brown or red, hen or rooster. Four are thin brown hens, 17 are hens, 14 are thin chickens, 4 are thin hens, 11 are thin brown chickens, 5 are brown hens, 3 are fat red roosters, 17 are thin or brown chickens. How many chickens does Mr. Brown have?

15. Consider the following information regarding three sets A, B, and C all of which are subsets of a set U. If $N(S)$ denotes the number of members of S, suppose that $N(A) = 14$, $N(B) = 10$, $N(A \cup B \cup C) = 24$ and $N(A \cap B) = 6$. Consider the following assertions:

    1. C has at most 24 members
    2. C has at least 6 members
    3. $A \cup B$ has exactly 18 members

    Which ones are true?