Problem Set 2, Handbook 2000 Problems

1. How many integers in the set $\{1, 2, 3, 4, \ldots, 360\}$ have at least one prime divisor in common with 360?

   Solution: 264

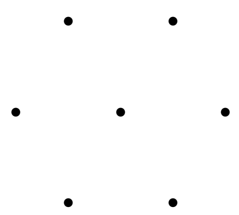
2. Use the digits 2 through 9, one per square, to maximize the value of

   $$\square\square\square\square + (\square\square \times \square\square).$$

   What is that maximum value?

   Solution: $15,932 = 9632 + 84 \times 75$.

3. How many equilateral triangles have all three vertices in the hexagonal lattice shown?

   Solution: 8

4. The product of the digits of a four-digit number is $6!$. What is the largest the number could be? What is the smallest it could be? How many such numbers are there?

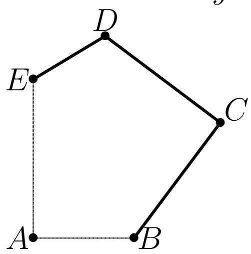
   Solution: largest: 9852, smallest: 2589, how many: 72.

5. Find all pairs of positive integers, (x, y) such that

   $$1 + 4x + 6y = xy.$$

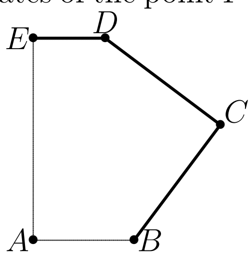
   Solution: Transform to $xy - 6y - 4x - 5^2 + 4 \cdot 6 = 0$. Then factor to get $(x - 6)(y - 4) = 25$. Finally, find six different pairs.

6. The pentagon $ABCDE$ with vertices $A = (0,0)$, $B = (7,0)$, $C = (13,8)$, $D = (5,14)$, and $E = (0,y)$ has perimeter 42. What is y?

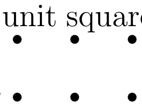


Solution: $y = 2$.

7. The pentagon P with vertices $A = (0,0)$, $B = (7,0)$, $C = (13,8)$, $D = (5,14)$, and $E = (0,14)$. A line L through the origin divides P into two quadrilaterals with equal perimeters. Find the coordinates of the point F where L meets $\overline{CD}$.



Solution: $6/10(5,14) + 4/10(13,8) = (8.20, 11.60)$.

8. There are eight unit squares that have two or more vertices in the 2 by 3 array of lattice points 
How many unit squares have at least two vertices in an m by n array of lattice points?

Solution: Its $(m-1)(n-1) + 2(n-1) + 2(m-1)$

9. Each of the six faces of a plastic cube is colored either red or green with equal probability. What is the probability that such a coloring results in a cube that has a vertex, all three of whose containing faces is the same color?

   Solution: $46/64 = 23/32$.

10. Three faces of a cube are randomly selected. What is the probability that they have a common vertex?

    Solution: $2/5$.

11. Find a number that differs by 1 from the sum of the squares of its digits.

    Solution: 35 and 75

12. A point is randomly selected from the triangle with vertices at $(0, 0)$, $(2, 0)$, and $(0, 3)$. What is the probability that the point is within one unit of $(0, 0)$? Express your answer in terms of π.

    Solution: $\pi/12$

13. A point is randomly selected from the rectangle with vertices at $(0, 0)$, $(2, 0)$, $(2, 3)$ and $(0, 3)$. What is the probability that the x-coordinate of the point is less than the y-coordinate?

    Solution: $2/3$.

14. Both $\odot$ and $*$ are in the set $\{+, \times, \div, -\}$, and

    $$(12 \odot 2) \div (9 * 3) = 2/9.$$

    Compute the value of $(8 \odot 4) \div (1 * 2)$.

    Solution: 1

15. Compute the value of $99^3 + 3 \cdot 99^2 + 3 \cdot 99$

    Solution: $100^3 - 1 = 999,999$.

16. There are several sets of three different numbers whose sum is 14 which can be chosen from $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. How many of these sets contain a 4?

    Solution: 3

17. Two circles of radius 1 are centered at $(4, 0)$ and $(-4, 0)$. How many circles contain exactly one point of each of the given circles and also the point $(0, 5)$?

    Solution: 4

18. How many integers in the range 500 to 999 have no consecutive identical digits. For example, 626 qualifies but 722 does not.

    Solution: 405

19. The function f is linear and satisfies $f(d+1) - f(d) = 3$ for all real numbers d. What is $f(3) - f(5)$?

    Solution: -6.