My Favorite Problems, 14
Harold B. Reiter
University of North Carolina Charlotte

This is the fourteenth of a series of columns of mathematics problems. I am soliciting future problems for this column from the readers of *M&I Quarterly*. I'm looking for problems with solutions that don't depend on highly technical ideas. Ideal problems should be **easily understood** and accessible to bright high school students. Their solutions should require a clever use of a well-known problem solving technique. Send your problems and solutions by email to me at `hbreiter@email.uncc.edu`. In general, we'll list the problems in one issue and their solutions in the next issue.

14.1 Find six different nonzero decimal digits a, b, c, d, e, f so that $\frac{a}{b} + \frac{c}{d} + \frac{e}{f} < 1$ and the sum is as large as possible.

14.2 What is the volume of the polyhedron Q defined by $|z-3| + |x-y| + |x+y| + |x| + |y| \leq 6$?

14.3 Let C denote the 16-element set $\{(a_1, a_2, a_3, a_4) \mid a_i \in \{0, 1\}, i = 1, 2, 3, 4\}$ in Euclidean space E_4. Let T denote the set of all triangles all of whose vertices belong to C. How many members of T are acute? How many members of T are right triangles? How many members of T are obtuse?

Problems from My Favorite Problems, 13, with solutions.

13.1 The three points $(4, 14, 8, 14)$, $(6, 6, 10, 8)$ and $(2, 4, 6, 8)$ are vertices of a cube in 4-space. Find the center of the cube.

Solution: Let $A = (4, 14, 8, 14)$, $B = (6, 6, 10, 8)$ and $C = (2, 4, 6, 8)$, and let d denote the (Euclidean) distance function. Then $d(A, B) = \sqrt{54}$, $d(A, C) = \sqrt{72}$, and $d(B, C) = \sqrt{18}$, so ABC is a right triangle with hypotenuse AC, by the converse of the Pythagorean Theorem. Since the distances are in the ratio $1 : \sqrt{3} : 2$ the segment AC connects vertices at opposite corners of the cube. Therefore, the center of the cube is the midpoint of this segment, $(3, 9, 7, 11)$.

13.2 Let $n \geq 1$ be fixed. Suppose n points are placed at random on a circle. Let $P(n)$ denote the probability that all n points lie on the same side of some diameter? In particular, find $P(2)$ and $P(3)$.

Solution: Let us first inscribe a regular $2m+1$-gon in this circle where $m \geq n-1$ and m is fixed. We will calculate $(*)$ the probability if we choose n of these $2m+1$ vertices at random then these n points will lie on the same side of some diameter.

Now the number of different ways that n vertices can be chosen from these $2m+1$ vertices so that these n vertices will lie on the same side of some diameter equals $(2m+1)\binom{n-1}{m}$. To see this, orient the circle in the counter clock wise direction as shown. Therefore, if n points lie on the same side of a diameter of the oriented circle, then we can single out a first member of these n points and call it $\overline{0}$. Thus, the $\overline{0}$ can be chosen in $2m+1$ different ways, and once $\overline{0}$ is chosen the other $n-1$ points can be chosen in $\binom{n-1}{m}$ different ways.

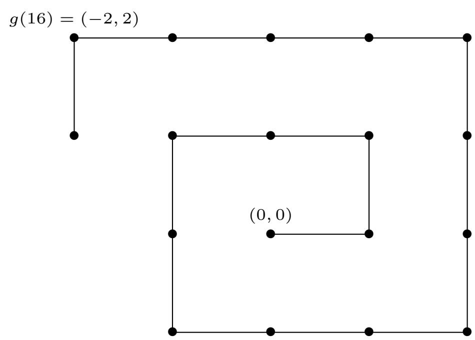
The probability required in $(*)$ equals

$$\frac{(2m+1)\binom{n-1}{m}}{\binom{n}{2m+1}} = \frac{n \cdot [m(m-1)(m-2)\cdots(m-(n-2))]}{2m(2m-1)(2m-2)\cdots(2m-(n-2))}.$$

The solution to problem (a) is

$$\lim_{m\to\infty} \frac{n \cdot [m(m-1)(m-2)\cdots(m-(n-2))]}{2m(2m-1)(2m-2)\cdots(2m-(n-2))} = \frac{n}{2^{n-1}}.$$

13.3 The bug is back, again crawling around the plane at a uniform rate, one unit per minute. He starts at the origin at time 0 and crawls one unit to the right, arriving at $(1, 0)$, turns $90°$ left and crawls another unit to $(1, 1)$, turns $90°$ left again, and crawls two units. He continues to make $90°$ left turns as shown in the figure. (The path of the bug establishes a one-to-one correspondence between the non-negative integers and the integer lattice points of the plane.) Let $g(t)$ denote the position in the plane after t minutes, where t is an integer. Thus, for example, $g(0) = (0, 0)$, $g(6) = (-1, -1)$, and $g(16) = (-2, 2)$. Does there exist an integer t such that $g(t)$ and $g(t + 23)$ are exactly 17 units apart? If so, find the smallest such t.



Solution: The path from $g(t)$ to $g(t+23)$ must involve exactly one turn. The only solutions to $u^2 + (23 - u)^2 = 17^2$ are $u = 8$ and $u = 15$. The first straight edge of length 15 occurs on the left to right segment from $g(210) = (-7, -7)$ to $g(225) = (8, -7)$. Going back 8 units to $g(202) = (-7, 1)$, we see that

$$D((-7, 1), (8, -7)) = \sqrt{15^2 + 8^2} = 17.$$

Incidentally, the function g is given by

$$g(t) = \begin{cases} (t - 4n^2 - 3n, -n) & \text{if } 2n(2n+1) \le t < (2n+1)^2 \\ (n+1, t - 4n^2 - 5n - 1) & \text{if } (2n+1)^2 \le t < (2n+1)(2n+2) \\ (4n^2 - n - t, n) & \text{if } 0 < (2n-1)(2n+1) \le t < (2n)^2 \\ (-n, 4n^2 + n - t) & \text{if } 0 < (2n)^2 \le t < (2n)(2n+1) \end{cases}$$