My Favorite Problems, 16

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This is the sixteenth of a series of columns of mathematics problems. I am soliciting future problems for this column from the readers of *M&I Quarterly*. I'm looking for problems with solutions that don't depend on highly technical ideas. Ideal problems should be **easily understood** and accessible to bright high school students. Their solutions should require a clever use of a well-known problem solving technique. Send your problems and solutions by email to me at `hbreiter@email.uncc.edu`. In general, we'll list the problems in one issue and their solutions in the next issue.

16.1 (University of South Carolina Math Contest, 2006) Find a real function f with domain all of $\mathtt{R}$ except possibly two points such that

$$f(x) + f\left(\frac{1}{1-x}\right) = x.$$

16.2 The number $N = 37! = 1 \cdot 2 \cdot 3 \cdots \cdot 37$ is a 44-digit number. The first 33 digits are $K = 137637530912263450463159795815809$. In fact, $N = K \cdot 10^{11} + L \cdot 10^{8}$, where L is a 3-digit number. Find the three-digit number L.

Problems from My Favorite Problems, 15, with solutions.

15.1 (This beautiful problem is due to Andy Niedermaier, a graduate student at University of California San Diego) Consider a 10×10 grid of lights, each either on or off, which we denote using matrix notation $a_{i,j}$, where, for each $i = 1, 2, \ldots, 10$ and $j = 1, 2, \ldots, 10$, the entry in row i and column j is $a_{i,j}$ and its value is 0 or 1. We are allowed two types of *moves*. For each $1 \le u \le 8$ and $1 \le v \le 8$, we can change the status of all the lights $a_{i,j}$ for which both $u \le i \le u + 2$ and $v \le j \le v + 2$. This is called a *small block* move. The other type move is, for each $1 \le u \le 6$ and $1 \le v \le 6$, we can change the status of all the lights $a_{i,j}$ for which both $u \le i \le u+4$ and $v \le j \le v+4$. This is called a *large block* move. So essentially, we can change the status of all nine lights in each 3×3 subarray and of all the lights in each 5×5 subarray. Is it possible, beginning with the all on configuration, to achieve any configuration of lights?

Solution: There are 64 small block moves and 36 large block moves. Each sequence of moves can be reduced to a string(perhaps empty) of small block moves followed by a string of large block moves. This follows from three important facts. First, moves commute. The combined effect of two moves is to change the status of the symmetric difference of the sets they change. Second, each move is its own inverse, so a move need never be listed more than once. Third, the compositions of moves is associative because the taking of symmetric differences is associative. Thus, every sequence of moves can be reduced to a 100-vector of zeros and ones. Therefore, at most 2^{100} possible configurations can be achieved. But note that there are precisely 2^{100} such light configurations. We'll be done if we can show that two different vectors of moves give the same configuration of lights. Denote the small block moves by $A_{u,v}$, and the large block moves by $B_{u,v}$, consider the sequence $A_{11}A_{16}A_{61}A_{66}$. This sequence leave the 36 positions $\{a_{ij} \mid ((1 \le i \le 3) \text{ or } (6 \le i \le 8)) \text{ and } ((1 \le j \le 3) \text{ or } (6 \le j \le 8))\}$ changed. The sequence $B_{11}B_{14}B_{41}B_{44}$ also leaves the 36 positions $\{a_{ij} \mid ((1 \le i \le 3) \text{ or } (6 \le i \le 8)) \text{ and } ((1 \le j \le 3) \text{ or } (6 \le j \le 8))\}$ changed.

15.2 Solve the equation $\sqrt{5-x} = x^2 - 5$ over the reals.

Solution: The interesting part of this problem is the unusual use of the quadratic formula. Begin by squaring both sides to get $5 - x = x^4 - 10x^2 + 25$, which we can write as $5^2 - 5(2x^2 + 1) + x^4 + x = 0$. Think of this as a quadratic equation in 5! Then

$$5 = \frac{2x^2 + 1 \pm \sqrt{(2x^2+1)^2 - 4 \cdot (x^4 + x)}}{2}.$$

Working with just the radical, we have

$$\begin{aligned} \sqrt{(2x^2+1)^2 - 4 \cdot (x^4 + x)} &= \\ \sqrt{4x^4 + 4x^2 + 1 - 4x^4 - 4x} &= \\ \sqrt{4x^2 - 4x + 1} &= \\ \sqrt{(2x-1)^2} &= |2x - 1| \end{aligned}$$

These two values reduce to $x^2 - x + 1$ and $x^2 + x$, so we must solve the two quadratics $5 = x^2 - x + 1$ and $5 = x^2 + x$. Each of these provides one real root and one extraneous root. Of course, the use of the quadratic formula simply disguises the factorization of the quartic:

$$x^4 - 5x^2 + x + 20 = (x^2 - x - 4)(x^2 + x - 5).$$