My Favorite Problems, 17

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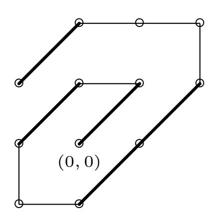
This is the seventeenth of a series of columns of mathematics problems. I am soliciting future problems for this column from the readers of *M&I Quarterly*. I'm looking for problems with solutions that don't depend on highly technical ideas. Ideal problems should be **easily understood** and accessible to bright high school students. Their solutions should require a clever use of a well-known problem solving technique. Send your problems and solutions by email to me at `hbreiter@email.uncc.edu`. In general, we'll list the problems in one issue and their solutions in the next issue.

17.1 Let $f(x) = x^2 + 8x + 12$. Find all real solutions of the equation $f(f(f(f(f(x))))) = 0$.

17.2 A bug starts from the origin in the plane and crawls to $(1, 1)$ after one minute. After another minute it crawls to $(0, 1)$. Consider the counterclockwise spiral path, shown below, starting at the origin. Each unit segment between lattice points take exactly one minute to traverse and each diagonal segment of length $\sqrt{2}$ also takes one minute.



(a) Where is the bug after exactly 2008 minutes?

(b) How many minutes does it take for the bug to get to the ordered pair $(19, 99)$?

Problems from My Favorite Problems, 16, with solutions.

16.1 (University of South Carolina Math Contest, 2006) Find a real function f with domain all of $\mathbb{R}$ except possibly two points such that

$$f(x) + f\left(\frac{1}{1-x}\right) = x.$$

Solution: Note that the given condition implies

$$a. f\left(\frac{1}{1-x}\right) + f\left(\frac{x-1}{x}\right) = \frac{1}{1-x},$$

and

$$b. f\left(\frac{x-1}{x}\right) + f(x) = \frac{x-1}{x}.$$

Taking the sum of the given equation with b and subtracting a yields $2f(x) = x - \frac{1}{1-x} + \frac{x-1}{x}$, which is equivalent to

$$f(x) = \frac{-x^3 + x - 1}{2x(1-x)}.$$

16.2 The number $N = 37! = 1 \cdot 2 \cdot 3 \cdots 37$ is a 44-digit number. The first 33 digits are $K = 137637530912263450463159795815809$. In fact, $N = K \cdot 10^{11} + L \cdot 10^8$, where L is a 3-digit number. Find the three-digit number L.

Solution: $37! = 13,763,753,091,226,345,046,315,979,581,580,902,400,000,000$. So the number L is 24. Let $L = \underline{abc}$. There are three solutions. One depends on the fact that $1001 = 7 \cdot 11 \cdot 13$. Write 37! in base 1000, so that each digit is of the form uvw, where u, v, and w are decimal digits. Thus $37! = 13 \cdot (1000)^{14} + 763 \cdot (1000)^{13} + \cdots + \underline{9ab} \cdot (1000)^3 + \underline{c00} \cdot 1000^2$. But since $1000^n = (1001-1)^n = 1001^n - n \cdot 1001^{n-1} + \cdots + (-1)^n$, by the binomial theorem, we can see that only the last term fails to be a multiple of 1001. This means that $uvw \cdot 1000^n \equiv (-1)^n \pmod{1001}$. Thus $37! \equiv 13 + 753 + 226 + 46 + 979 + 580 + 100c - (763 + 91 + 345 + 315 + 581 + 900 + 10a + b \equiv -398 + 100c - 10a - b \equiv 0 \pmod{1001}$. This can happen only when $c = 4, a = 0$ and $b = 2$, so $L = 24$.

The second method depends upon the fact that $999 = 27 \cdot 37$, so 37! is a multiple of 999. Once again, express the number 37! in base 1000, where $L = \underline{abc}$. Since $1000 \equiv 1 \pmod{999}$, the number 37! is the sum of its base 1000 digits, mod 999. Hence, $37! \equiv 13 + 763 + 753 + 091 + 226 + 345 + 046 + 315 + 979 + 581 + 580 + \underline{9ab} + \underline{c00} + 000 + 000 \equiv 5592 + \underline{9ab} + \underline{c00} \equiv 0 \pmod{999}$. The only way this can happen is $5592 + \underline{9ab} + \underline{c00} = 6 \cdot 999 = 5994$, from which it follows that $\underline{ab} + \underline{c00} = 402$, which means $c = 4, a = 0$ and $b = 2$.

The third and easiest method to understand uses brute force to get the rightmost digit, and then mod 9 and mod 11 arithmetic to find the other two digits.