My Favorite Problems, 18

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This is the eighteenth of a series of columns of mathematics problems. I am soliciting future problems for this column from the readers of *M&I Quarterly*. I'm looking for problems with solutions that don't depend on highly technical ideas. Ideal problems should be **easily understood** and accessible to bright high school students. Their solutions should require a clever use of a well-known problem solving technique. Send your problems and solutions by email to me at `hbreiter@email.uncc.edu`. In general, we'll list the problems in one issue and their solutions in the next issue.

18.1 Construct a rectangle by putting together nine squares with sides equal to 1, 4, 7, 8, 9, 10, 14, 15 and 18.

18.2 Suppose $(S, 0, +)$ is a finite Abelian group on the set S, and $\cdot$ is a commutative binary operator on S. Also, suppose $(S, 0, +)$ distributes over $(S, \cdot)$. That is, $\forall a, b, x \in S,\ x + (a \cdot b) = (x + a) \cdot (x + b)$.

(a) Show that $|S|$ is odd.

(b) Also, given $(S, 0, +)$, find all binary operators $\cdot$ that satisfy these conditions.

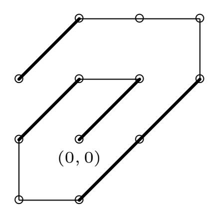
18.3 Consider the $a \times b \times c$ rectangular box built from abc unit cubes, where a, b, and c are positive integers. How many paths of length $a + b + c$ are there from a fixed corner of the box to the corner farthest away along edges of the unit cubes that stay on the surface of the box?

Problems from My Favorite Problems, 17, with solutions.

17.1 Let $f(x) = x^2 + 8x + 12$. Find all real solutions of the equation $f(f(f(f(f(x))))) = 0$.

Solution: A. Note that $f(x) = (x+4)^2 - 4$. Then $f(f(f(f(f(x))))) = (x+4)^{32} - 4$ and the answer is $x = -4 \pm 4^{\frac{1}{32}}$.

17.2 A bug starts from the origin in the plane and crawls to $(1,1)$ after one minute. After another minute it crawls to $(0,1)$. Consider the counterclockwise spiral path, shown below, starting at the origin. Each unit segment between lattice points take exactly one minute to traverse and each diagonal segment of length $\sqrt{2}$ also takes one minute.



(a) Where is the bug after exactly 2008 minutes?

Solution: Look at the time required to get to the points $(n,0)$ along the positive x-axis: 6 minutes to $(1,0)$, 18 minutes to $(2,0)$, $6 \cdot 6$ minutes to $(3,0)$. This suggests and we can prove that the time required to get to $(n,0)$ is $6 \cdot T_n = 6 \cdot \binom{n+1}{2} = 6 \cdot \frac{(n+1)n}{2}$. To see where the bug is after 2008 minutes, we must find the largest triangular number less that $2008/6 = 334.\overline{6}$. Now $\binom{26}{2} = 325 < 334$, so after $6 \cdot 325 = 1950$ minutes, the bug is at $(25,0)$. Now, 26 minutes later, the bug is at $(26,26)$. Yet another 26 minutes later, the bug is at $(0,26)$, and finally 6 minutes later, after $1950+26+26+6 = 2008$ minutes, the bug is at the position $(-6,20)$.

(b) How many minutes does it take for the bug to get to the ordered pair $(19,99)$?

Solution: It take the bug $\binom{99}{2} = 4851$ minutes to get to $(98,0)$, so it takes $4851 + 99$ minutes to get to $(99,99)$ and another 80 minutes to get to $(19,99)$.