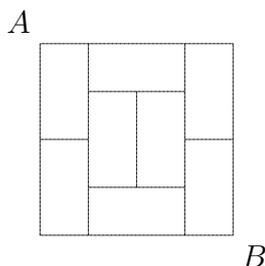


My Favorite Problems, 2
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This is the second of a series of columns about problems. I am soliciting problems from the readers of *M&I Quarterly*. Mathematical problems and challenges have enlivened my life over many years. In this column, I hope to share some of the 'ah ha's' I've enjoyed with my students and friends. I'd like to share your 'ah ha's' too. I'm looking for problems with solutions that don't depend on highly technical ideas. Ideal problems should be **easily understood** and accessible to bright high school students. Their solutions usually require a clever use of a well-known problem solving technique. For example, double counting, the principle of inclusion/exclusion, the pigeonhole principle, and Pick's theorem. Submitted problems need not be original. However, if the problem appeared in a contest, I want to acknowledge the contest. And, of course, if the name of the creator is available, that should be included with the problem. If you have a few problems whose solutions provoke you to say 'ah ha', please share them with *M&IQ* readers. Send your problems and solutions by email to me at hbreiter@email.uncc.edu. In general, we'll list the problems in one issue and their solutions in the next issue.

- 2.1 Longest path problem. Each rectangle in the diagram is 2×1 . What is the length of the longest path from A to B that does not retrace any part of itself? Prove that your answer is the best possible.



- 2.2 This captivating problem is due to John Conway, Princeton University. The Wizard Problem. Two wizards get on a bus, and one says to the other 'I have a positive number of children each of which is a positive number of years old. The sum of their ages is the number of this bus and the product of their ages is my age'. To this the second wizard replies 'perhaps if you told me your age and the number of children, I could work out their individual ages'. The first wizard then replies 'No, you could not.' Now the second wizard says 'Now I know your age'. What is the number of the bus? Note: Wizards reason perfectly, can have any number of children, and can be any positive integer years old. Also, consider the same problem but with the additional assumption that the children are all different ages.

The solutions to the problems in column 1 follow.

1.1 Dinner Bill Splitting Problem.

A dinner bill for s dollars and t cents is to be split as evenly as possible between two couples, where s and t are two-digit numbers. Since t is odd, the split cannot be exact. However, it turns out that twice t dollars and s cents differs by just one cent from s dollars and t cents. Find s and t . What can you say about s and t if we remove the restriction that they are two-digit numbers.

Solution: First, we'll state the problem in a more precise way. A dinner bill for s dollars and t cents is to be split as evenly as possible between two couples, where s and t are two-digit numbers. Since t is odd, the split cannot be exact. However, it turns out that twice t dollars and s cents differs by just one cent from s dollars and t cents. Find s and t . Symbolically, this means that $|s \cdot t - 2 \cdot t \cdot s| = 0.01$ where s and t are two digit numbers. Let's convert this to an integer equation by multiplying by 100:

$$|100s + t - 2(100t + s)| = 1.$$

This is equivalent to $|98s - 199t| = 1$ which can be interpreted as the two equations $98s - 199t = 1$ or $98s - 199t = -1$. Before continuing with the problem at hand, let's consider another problem whose solution will propel us forward solving the one at hand. The Decanting Problem is a liquid measuring problem that begins with two ungraduated decanters whose integer capacities a and b are given. The problem is to determine the smallest amount of liquid that can be measured and how such amount can be obtained, by a process of filling, pouring, and dumping. Specifically, there are three actions we can take:

- (a) fill an empty decanter,
- (b) dump out a full decanter, and
- (c) pour from one decanter to the other until either the receiving decanter is full or the poured decanter is empty.

Let's look at an easy one first. Let $a = 3$ and $b = 5$. A little thought and we see that we can fill the 3 unit decanter twice, and dump the 5 unit decanter once to get 1 unit of liquid. Specifically, $2 \cdot 3 - 1 \cdot 5 = 1$. Next, suppose the decanters have capacities 5 units and 7 units. A little experimentation leads to the conclusion that 1 unit of water can be obtained by filling the 5 unit decanter 3 times, pouring repeatedly from the 5 unit to the 7 unit decanter and dumping out the 7 unit decanter twice. A finite state diagram is helpful to follow the procedure:

$$(0, 0) \implies (5, 0) \implies (0, 5) \implies (5, 5) \implies (3, 7) \implies (3, 0) \implies (0, 3) \implies (5, 3) \\ (5, 3) \implies (1, 7) \implies (1, 0),$$

where the notation (x, y) means the 5-unit container has x units of liquid and the 7-unit container has y units.

Notice that the procedure includes 3 fills and 2 dumps, with fills and dumps alternating and separated by 4 pours. An arithmetic equation representing this is

$$3 \cdot 5 - 2 \cdot 7 = 1.$$

Notice that not only does the arithmetic equation follow from the state diagram, the reverse is also true. That is, given the arithmetic equation, it is an easy matter to construct the state diagram. In the next example, the least amount that can be measured is not 1. Let the decanters have sizes 15 and 99. Before reading on, can you see why it is impossible to obtain exactly one unit of water? An equation can be obtained for any sequence of moves. Such an equation is of the form

$$15x + 99y = z$$

where exactly one of the integers x and y is negative, and z is the amount obtained. Now notice that the left side is a multiple of 3, so the right side must be also. Thus the least positive amount that can be measured is 3 units. One can also reason as follows: each fill adds a multiple of 3 units of water, each pour leaves the number unchanged, and each dump removes a multiple of three units, so the amount on hand at each stage is a multiple of 3.

In fact, the answer is that the least amount that can be measured is the greatest common divisor of the two decanter sizes, and the Euclidean algorithm tells us how to proceed. Suppose $c = GCD(a, b)$. The Euclidean algorithm yields a solution to

$$c = ax + by$$

where x and $|y|$ are integers exactly one of which is positive and, except in trivial cases, the other is negative. For convenience, we assume x is positive. Then the solution to the decanting problem is to fill the a capacity decanter x times, repeatedly pouring its contents into the b unit decanter. The b unit decanter will be dumped y times, so the total water on hand at the end is the difference $ax - by = c$.

Let's look at another specific example. Again we use the Euclidean Algorithm to solve the decanting problem. There are two stages. The first stage is a sequence of divisions. The second is a sequence of 'unwindings'. For this example, let the decanter sizes be $a = 257$ and $b = 341$. Use the division algorithm to get $341 = 1 \cdot 257 + 84$. Then divide 257 by 84 to get $q = 3$ and $r = 5$. That is $257 = 3 \cdot 84 + 5$. Continue dividing until the quotient q becomes 0. Thus 84 divided by 5 yields $84 = 16 \cdot 5 + 4$. Finally, divide 5 by 4 to get $5 = 1 \cdot 4 + 1$. This completes the first stage. Now to unwind, start with the final division scheme write $1 = 5 - 1 \cdot 4$. Then replace the 4 with $84 - 16 \cdot 5$ to get $1 = 5 - 1(84 - 16 \cdot 5)$. This is equivalent to $1 = 17 \cdot 5 - 1 \cdot 84$. Check this to be sure. Then replace 5 with $257 - 3 \cdot 84$ to get

$$1 = 17 \cdot (257 - 3 \cdot 84) - 1 \cdot 84,$$

i.e., $1 = 17 \cdot 257 - 52 \cdot 84$. Finally, replace 84 with $341 - 257$ to get $1 = 17 \cdot 257 - 52(341 - 257)$, which we can rewrite as

$$1 = 69 \cdot 257 - 52 \cdot 341.$$

Thus, the solution to the decanting problem is to measure out 1 unit of water by filling the 257 unit decanter 69 times, repeatedly pouring its contents into the 341 unit decanter, and, in the process, dumping out the 341 unit decanter 52 times.

Now back to the bill splitting problem. Imaging that we have two decanters with capacities $a = 199$ and $b = 98$. Notice that $GCD(199, 98) = 1$. As we did above, we can use the Euclidean algorithm to find numbers x and y satisfying $199x + 98y = 1$ where exactly one of the numbers x, y is negative. We do this by dividing repeatedly. First, 98 into 199 yields

$199 = 2 \cdot 98 + 3$, Then 3 into 98 yields $98 = 32 \cdot 3 + 2$ and finally we can write $1 = 3 - 2$. Next we go to the unwinding stage.

$$\begin{aligned}
 1 &= 3 - 2 \\
 &= 3 - (98 - 32 \cdot 3) \\
 &= 3 - 98 + 32 \cdot 3 \\
 &= 33 \cdot 3 - 1 \cdot 98 \\
 &= 33(199 - 2 \cdot 98) - 98 \\
 &= 33 \cdot 199 - 66 \cdot 98 - 98 \\
 &= 33 \cdot 199 - 67 \cdot 98
 \end{aligned}$$

Thus, we have the values $s = 67$ and $t = 33$. Indeed, $2 \cdot 33.67 - 67.34 = -1$.

1.2 The 7-11 problem. A man goes into a convenience store, picks out four items, and goes to check out. The clerk tells him that her cash register is broken, and she will use her calculator. She proceeds to process the four amounts, and says, “that will be \$7.11”. “Wait a minute”, he protests, “you multiplied the prices together”. She promptly repeats the calculation, this time adding the four amounts, and exclaims, “there, you owe \$7.11, just as I said.” (There is no tax on food in this state.) There are two questions. First, what is the name of the convenience store, and what are the four prices? Challenge: try this problem with only three items. You’ll have to change the \$7.11, of course. Then try the problem for just two items. There are lots of solutions. Find them all. Then try the 7-11 problem with three items and a total bill of \$8.25. Find some other total cost that could be used to solve the three item 7-11 problem.

Solution: The four prices are \$1.25, \$1.20, \$1.50 and \$3.16. To see how to get these numbers, let $\underline{x}, \underline{y}, \underline{u}$, and \underline{v} denote the four prices, in dollars. Then $\underline{xyuv} = 7.11$ and $\underline{x} + \underline{y} + \underline{u} + \underline{v} = 7.11$. To eliminate the fractional part, multiply each of the unknowns and rename to get $x = 100\underline{x}, y = 100\underline{y}, u = 100\underline{u}$, and $v = 100\underline{v}$. Thus we have $xyuv = 10^8 \cdot 7.11$ and $x + y + u + v = 711$. Factor the former to get $xyuv = 711 \cdot 10^6 = 2^6 \cdot 3^2 \cdot 5^6 \cdot 79$. It follows that exactly one of x, y, u, v must be a multiple of 79. For convenience, let’s say its v . Then $v = 79, 158, 237$, or 316. We start by examining the last choice, 316. In this case, $xyu = 711 \cdot 10^6 \div 316 = 2^4 \cdot 3^2 \cdot 5^6$ and $x + y + u = 711 - 316 = 395$. Note that $\sqrt[3]{2^4 3^2 5^6} = 50\sqrt[3]{18} > 125$, so the sum $x + y + u$ must be at least $3 \cdot 125 = 375$. Therefore we try to minimize $x + y + u$ subject to $xyu = 2^4 3^2 5^6$. This occurs when we choose x, y , and u as close together as possible. Hence, let $x = 5^3 = 125, y = 2^3 \cdot 3 \cdot 5 = 120$ and $u = 2 \cdot 3 \cdot 5^2 = 150$. Thus the four prices are $\underline{x} = \$1.25, \underline{y} = \$1.20, \underline{u} = \$1.50$ and $\underline{v} = \$3.16$. In the three item problem, we have the following: 6.00 (1, 2, 3); 8.25 (.75, 2, 5.5); 9.00 (.5, 4, 4.5); 10.80 (.4, 5, 5.4).

- 1.3 (1997 American High School Math Examination, problem 29) A number is called 7-special if its decimal representation consists of only two digits, 0 and 7. For example, $7/99 = 0.\overline{07}$ and 7.707 are such numbers. It is possible to write 1 as a sum of 7-special numbers. If so, what is the fewest number of 7-special numbers whose sum is 1?

Solution: Divide 1 by 7 to get $1/7 = \overline{.142857}$. If we can write $1/7$ as a sum of number that require on the digits 0 and 1, then we can multiply all these summands by 7 to write 1 as a sum of 7-special numbers. Clearly we can write $1/7$ as a sum of 8 numbers that require only the digits 0 and 1. Try $a = \overline{.111111}$, $b = \overline{.011111}$, $c = \overline{.010111}$, $d = \overline{.010111}$, $e = \overline{.000111}$, $f = \overline{.000101}$, $g = \overline{.000101}$ and $h = \overline{.000100}$. Replacing all the 1's by 7's results in eight 7-special numbers whose sum is 1.