My Favorite Problems, 25

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This is column twentyfive of a series of columns of mathematics problems. I am soliciting future problems for this column from the readers of *M&I Quarterly*. I'm looking for problems with solutions that don't depend on highly technical ideas. Ideal problems should be **easily understood** and accessible to bright high school students. Their solutions should require a clever use of a well-known problem solving technique. Send your problems and solutions by email to me at `hbreiter@uncc.edu`. In general, we'll list the problems in one issue and their solutions in the next issue.

25.1 Let $a_1 = 1$ and for each $n \geq 1$, define a_{n+1} as follows:

$$a_{n+1} = 2n - 1 + a_n \cdot 10^{1+\lfloor \log(2n-1) \rfloor}.$$

How many of the numbers $a_1, a_2, \ldots, a_{1000}$ are multiples of 9?

25.2 Find, with proof, the largest n for which $n = 7S(n)$, where $S(n)$ denotes the sum of the (decimal) digits of the integer n.

25.3 Football teams score $1, 2, 3$, or 6 points at a time. They can score 1 point (point-after-touchdown) only immediately after scoring 6 points (a touchdown). A scoring sequence is a sequence of numbers $1, 2, 3, 6$, where all the 1's are immediately preceded by 6. Both $2, 6, 1, 3, 2$ and $2, 2, 6, 1, 3$ are scoring sequences with *Value* 14. How many scoring sequences have Value 14?

Problems from My Favorite Problems, 24, with solutions.

24.1 After each touchdown (and extra point), and also after each field goal, Sir Purr does a pushup for each Panther point on the scoreboard. For example, if the Panthers scored 3, 7,and 3 points in that order, Sir Purr would do a total of $3 + 10 + 13 = 26$ pushups. Against the Steelers, Sir Purr had to do exactly 100 pushups. How many points could the Panthers have scored in the game. Assume they scored points in groups of 3 and 7 only.

Solution: 25 and 29. This was the tiebreaker question at the 2010 Carolina Panthers' Number Crunch Mathematics Competition. The total number of pushups is $3f + 7t$ for some integers f and t where $f + t$ is a triangular number (ie, $1, 3, 6, 10, 15, 21, \ldots$). In order that $3f + 7t = 100$, $100 - 7t$ must be a multiple of 3. Thus $t = 1, 4, 7, 10$, or 13. The corresponding values of f are $31, 24, 17, 10$, and 3. Only one of these $t = 4, f = 24$ has a triangular sum. To see that $t = 4, f = 24$ can lead to 100 pushups, consider both $3, 3, 3, 7, 3, 3, 3$ and $3, 3, 3, 3, 7, 3, 7$. Thus the Panthers could have scored either 25 points: $3 + 6 + 9 + 16 + 19 + 22 + 25 = 100$ or 29 points: $3 + 6 + 9 + 12 + 19 + 22 + 29 = 100$.

24.2 How many ways can 1000 be expressed as a sum of powers of 2 if at most 2 of each power is allowed? For example, $8 = 4 + 4 = 4 + 2 + 2 = 4 + 2 + 1 + 1$, so there are four ways to write 8 as such a sum.

Solution: Let $G(n)$ denote the number of ways to write n as such a sum. Then $G(2^k) = G(2^{k-1}) + 1 = k + 1$. We can represent n is pseudo-binary, that is, place value with digits 0, 1, and 2 allowed. Also, if n is odd, then the rightmost digit in the representation is 1, and we have $G(2n + 1) = G(n)$ since each representation of n can be appended with a 1 to get a representation of $2n + 1$. Similarly, $G(2n) = G(n) + G(n - 1)$. This follows from the fact that if $n = a_k a_{k-1} \ldots a_0$ then $2n = a_k a_{k-1} \ldots a_0 0$; that is, append 0 to the right end of any representation of n. Also, if $n - 1 = a_k a_{k-1} \ldots a_0$, then $2n = a_k a_{k-1} \ldots a_0 2$; that is, append a 2 at the right end of any representation is $n - 1$. The effect is to double and add 2. So we can work our way from 1000 to powers of 2, and when we do, we get $G(1000) = 33$.

24.3 Use all ten digits exactly once to build three prime numbers with the least sum.

Solution: Solution by C. Kevin Chen. $1069 + 457 + 283 = 1809$. First note that our three numbers must consist of a four-digit and two three-digit numbers. Then we must put the 1 in the thousands place and the zero in the hundreds place. This leave only the digits $9, 7$, and 3 for units digits. Thus the 2 and the 4 are the least possible hundreds digits, leaving the $6, 5$, and 7 as tens digits. To see that this can, in fact, be done requires a little shuffling around.