My Favorite Problems, 26

Harold B. Reiter
University of North Carolina Charlotte

This is column twentysix of a series of columns of mathematics problems. I am soliciting future problems for this column from the readers of *M&I Quarterly*. I'm looking for problems with solutions that don't depend on highly technical ideas. Ideal problems should be **easily understood** and accessible to bright high school students. Their solutions should require a clever use of a well-known problem solving technique. Send your problems and solutions by email to me at `hbreiter@uncc.edu`. In general, we'll list the problems in one issue and their solutions in the next issue.

26.1 Determine the unique pair of real numbers (x, y) that satisfy the equation

$$(4x^2 + 6x + 4)(4y^2 - 12y + 25) = 28.$$

26.2 Use the digits of the set $\{1, 3, 4, 6\}$ and some of the arithmetic operations $+, \times, -, \div$ to create the number 24.

26.3 John and Betty are the first to arrive at the game. They take their seats k and $k + 1$, and Betty notices that the seat numbers to her right, $1, 2, 3, \ldots, k - 1$ have a sum that is exactly one-seventh of the sum of the seat numbers to John's left. What assumptions do you need to assert that the solution is unique? What seat numbers do they occupy?

Problems from My Favorite Problems, 25, with solutions.

25.1 Let $a_1 = 1$ and for each $n \geq 1$, define a_{n+1} as follows:

$$a_{n+1} = 2n - 1 + a_n \cdot 10^{1+\lfloor \log(2n-1) \rfloor}.$$

How many of the numbers $a_1, a_2, \ldots, a_{1000}$ are multiples of 9?

Solution: There are 333 such multiples. The sum of the digits of a_n is n^2(modulo 9), so every third number in the sequence is a multiple of 9. In other words, a_{3k} is a multiple of 9 for every $k = 1, 2, \ldots, 333$.

25.2 Find, with proof, the largest n for which $n = 7S(n)$, where $S(n)$ denotes the sum of the (decimal) digits of the integer n.

Solution: This problem was inspired by a Problem of the Month proposal of Professor Bela Bajnok of Gettysburg College. First note that n cannot be a four-digit or larger number. If so, we'd have $\underline{abcd} = 7(a + b + c + d) \leq 7 \cdot 36 < 1000$, a contradiction. Now if $n = \underline{abc}$, and $n = 7S(n)$, then we have $100a + 10b + c = 7a + 7b + 7c$, which is equivalent to $93a + 3b = 6c$ and to $31a + b = 2c$. Only $a = 0$ can work here since c is a digit. Solving $b = 2c$ for the largest possible values of b and c yields $b = 8, c = 4$, so $n = 84$ is the largest integer with the desired property.

25.3 Football teams score $1, 2, 3,$ or 6 points at a time. They can score 1 point (point-after-touchdown) only immediately after scoring 6 points (a touchdown). A scoring sequence is a sequence of numbers $1, 2, 3, 6$, where all the 1's are immediately preceded by 6. Both $2, 6, 1, 3, 2$ and $2, 2, 6, 1, 3$ are scoring sequences with *Value* 14. How many scoring sequences have Value 14?

Solution: 54. We condition on the number of touchdowns, which is $2, 1$ or none. With 2 touchdowns the scoring sequences with Value 14 are $6161, 266, 626, 662$. With one touchdown there are three patterns depending on the number of field goals. With no field goals, we have 62222 and four others. With one field goal, the pattern is 61322, which we view as a permutation of four letters (the 6 and 1 stuck together is a letter. There are 12 permutations because the two 3's are indistinguishable. Then there is the pattern 6332 with two field goals. There are 12 of these. Finally, with no touchdowns, we have the patterns 332222, 33332, and 2222222. There are 15 of the first type, 5 of the second, and 1 of the third. So we have $4 + (12 + 5 + 12) + (15 + 5 + 1) = 54$ scoring sequences with Value 14.