My Favorite Problems, 3

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This is the third of a series of columns about problems. I am soliciting problems from the readers of *M&I Quarterly*. I'm looking for problems with solutions that don't depend on highly technical ideas. Ideal problems should be **easily understood** and accessible to bright high school students. Their solutions usually require a clever use of a well-known problem solving technique. For example, double counting, the principle of inclusion/exclusion, the pigeonhole principle, and Pick's theorem. Submitted problems need not be original. However, if the problem appeared in a contest, I want to acknowledge the contest. And, of course, if the name of the creator is available, that should be included with the problem. If you have a few problems whose solutions provoke you to say 'ah ha', please share them with *M&IQ* readers. Send your problems and solutions by email to me at `hbreiter@email.uncc.edu` . In general, we'll list the problems in one issue and their solutions in the next issue.

3.1 Linked Triangles. This beautiful problem is due to Sam Vandervelde, Greater Testing Concepts. Let A, B, C, D, E, and F be six points in 3-space, no four of which lie on the same plane. (I.e. six points in general position.) We say that triangles ABC and DEF are "linked" if exactly one of segments AB, AC, or BC intersects the interior of triangle DEF. This has the effect of linking the triangles, so that if the sides were made of thin metal rods it would be impossible to separate them. (Try drawing a picture!) The problem is to show that no matter how the six points are positioned in space (as long as no four are coplanar) it is always possible to split them into two sets of three points each so that the two resulting triangles (with the points in each set as vertices) are linked.

3.2 This pair of problems came to me from Wen-Hsien Sun. Version A was on the Fifth Po Leung Kuk Primary Math World Contest of Taiwan.
Version A. A teacher whispers a positive integer p to student P, a positive integer q to student Q, and a positive integer r to student R. The students don't know one another's numbers but they know the sum of the three numbers is 14. The students make the following statements:

(a) P says 'I know that Q and R have different numbers'.

(b) Q says 'I already knew that all three of our numbers are different'.

(c) R says 'Now I know all three of our numbers'.

What is the product of the three numbers?

Version B. A teacher whispers a positive integer p to student P, q to student Q, and r to student R. The students don't know one another's numbers but they know the sum of the three numbers is 14. The students make the following statements:

(a) P says 'I know that Q and R have different numbers'.

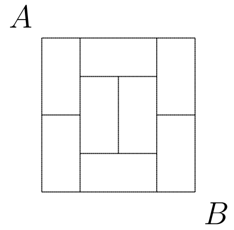
(b) Q says '<u>Now I know</u> that all three of our numbers are different'.

(c) R says 'Now I know all three of our numbers'.

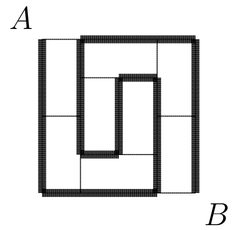
Assume all three students reason perfectly. What is the product of the three numbers?

Solutions to problems in Favorite Problems 2.

2.1 Longest path problem. Each rectangle in the diagram is 2×1. What is the length of the longest path from A to B that does not retrace any part of itself? Prove that your answer is the best possible.



Solution: We think of the grid as a graph, with 18 vertices, and 25 edges. The vertices have *degrees* 2 and 3. There are four vertices of degree 2 and 14 of degree 3. The edges also come in two types, those with length 2 (there are 7 of these) and those with length 1 (there are 18 of these). Since we are starting at A, only one of the edges adjacent to A can be part of the path.



The same is true for B. At all the other vertices, we can and must use exactly two edges. At the corners we have only two edges, but at all the other vertices, we must not use one of the incident edges. It is possible to choose a path so that the edges that are not used all have length 1. If every vertex belongs to the path, then exactly 17 edges can belong to the path, which means 8 edges do not belong. If all these edges have length 1, then the length of the path must be maximal, $L = 2 \cdot 7 + 18 \cdot 1 - 8 \cdot 1 = 14 + 18 - 8 = 24$. Note below that the path from A to B of length 24.

2.2 The Wizard Problem. Two wizards get on a bus, and one says to the other 'I have a positive number of children each of which is a positive number of years old. The sum of their ages is the number of this bus and the product of their ages is my age'. To this the second wizard replies 'perhaps if you told me your age and the number of children, I could work out their individual ages'. The first wizard then replies 'No, you could not.' Now the second wizard says 'Now I know your age'. What is the number of the bus? Note: Wizards reason perfectly, can have any number of children, and can be any positive integer years old. Also, consider the same problem but with the additional assumption that the children are all different ages.

Solution: Let's call a positive integer *ambiguous* if there are two different partitions of equal size of it into summands whose products are the same. For example, 14 is ambiguous because $14 = 3+3+8 = 2+6+6$ and $3 \cdot 3 \cdot 8 = 2 \cdot 6 \cdot 6 = 72$. A number is called *doubly ambiguous* if there are two pairs of partitions each of which has the same product, and these two products are different. For example, 13 is doubly ambiguous since $13 = 1 + 6 + 6 = 2 + 2 + 9$ and $1 \cdot 6 \cdot 6 = 2 \cdot 2 \cdot 9 = 36$ and at the same time $13 = 1 + 1 + 3 + 4 + 4 = 1 + 2 + 2 + 2 + 6$ and $1 \cdot 1 \cdot 3 \cdot 4 \cdot 4 = 1 \cdot 2 \cdot 2 \cdot 2 \cdot 6 = 48$. Now the first wizard's response to the second wizard's comment is equivalent to 'the bus number is ambiguous'. Note that if n is ambiguous, then so is $n + 1$ and if n is doubly ambiguous, then $n + 1$ is also doubly ambiguous. An examination of all partitions of 11 shows that 11 is not ambiguous. Thus every integer n at least 12 is ambiguous, and every integer bigger than or equal to 13 is doubly ambiguous. The second wizard's comment that he knew the age of the first wizard implies that the bus number is not doubly ambiguous. Thus the bus number must be 12.