My Favorite Problems, 5
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This is the fifth of a series of columns of mathematics problems. I am soliciting future problems for this column from the readers of *M&I Quarterly.* I'm looking for problems with solutions that don't depend on highly technical ideas. Ideal problems should be **easily understood** and accessible to bright high school students. Their solutions should require a clever use of a well-known problem solving technique.

5.1 *This beauty comes form the Spring 2001 Tournament of Towns contest.* You have three piles of stones containing 5, 49, and 51 stones. You can join any two piles together into one pile and you can divide any pile with an even number of stones into two piles of equal size. Can you ever achieve 105 piles, each with one stone?

5.2 You sit at a table that has some coins on it. Each one, of course, is either showing heads or tails. You are wearing a blindfold and thick gloves, so it is impossible for you to tell by sight or touch what each coin is showing. At the outset, you know how many coins are on the table, and how many are showing heads. You can do whatever you like to the coins–turn them over, etc. as long as all the coins end up back on the table. The question is, how can you divide the coins into two groups so that each has the same number of heads?

Problems from My Favorite Problems, 4 with solutions.

4.1 *This problem came to me from Stan Wagon, Macalester College, who credits Frank Rubin.* Suppose you have n keys on a circular key chain and wish to put a colored sleeve on each key so that it is identifiable by colors alone, without reference to its shape. For some n this can be done with fewer than n colors. For example, if you have 4 keys then you can place the colors on the keys as follows:

Red

Red *Green*

Blue

The top key is "the red key that is next to the green key across from the blue key;" and "the leftmost key is the red key that is adjacent to a blue one;" the other two are identifiable by their colors alone.

An identification scheme must work even if the key chain is flipped over, so one cannot use the words right and left in the identification scheme. Let $f(n)$ be the least number of colors necessary so that each key on a chain of n keys can be uniquely identified as explained above. Then $f(1) = 1, f(2) = 2$, and $f(3) = 3$. What is $f(123)$?

Solution: The function is given by $f(1) = 1, f(2) = 2, f(3) = 3, f(4) = 3, f(5) = 3, f(n) = 2$, for all $n \geq 6$. For $n = 3, 4, 5$, take one blue and one green and put them next to one another. That determines the orientation. Then you can count the number of red, starting say from the blue one. For $n \geq 6$, use three blue ones, two together and one separated by a red one. For example when $n = 7$, we get $BBRBRRR$, each of which is uniquely identified by its position. In particular, $f(123) = 2$.

4.2 For each ordered pair (j, k) of nonnegative integers for which the number $5^j 7^k$ is less that 10^{29}, there are at least three consecutive positive integers (perhaps four)$i, i+1$, and $i+2$ such that $2^i 5^j 7^k < 2^{i+1} 5^j 7^k < 2^{i+2} 5^j 7^k < 10^{30}$. There are more than one thousand such numbers $2^i 5^j 7^k$. Prove that in every 30-digit number of this form, some digit appears at least 4 times in its representation.

Solution: Let N denote any 30-digit number of the form $2^i 5^j 7^k$. Suppose that every digit appears exactly 3 times in the decimal representation of N. Then the sum of the digits is $3(0+1+2+\cdots+9) = 3 \cdot 45 = 135$ which is divisible by 9. This implies that N itself is divisible by 9. But the Fundamental Theorem of Arithmetic asserts that N has only one factorization into primes. The prime factors of N are $2, 5$ and 7, but not 3, so we have a contradiction.

4.3 Definition: An n-staircase is a grid of $\binom{n+1}{2}$ squares arranged so that column 1 has 1 square, column 2 has 2 squares, ..., and column n has n squares. *My friend Rick Armstrong in Mathematics and Computer Education* http://www.macejournal.org/ posed the question of how many square regions are bounded by the gridline of an n-staircase. My question for readers is this: How many rectangular regions are bounded by the gridlines of an n staircase?

Solution: The answer is that there are $\binom{n+3}{4}$ rectangular regions bounded by the gridlines of an n-staircase. *This solution was sent to me by James Rudzinski.* Let G_n denote the total number of rectangular regions in an n-staircase, $n \geq 1$. Also let the unit square that is the intersection of the longest column and the longest row of the n-staircase be called square A. See the 5-staircase below. First, note that if we remove either the longest column or the longest row from the n-staircase we would get an $(n-1)$-staircase. The number of rectangular regions in either of these subsections of the n-staircase would be G_{n-1}. If we count the rectangular regions in both subsections then we will get a double-count of all of the rectangular regions in the intersection of the two $(n-1)$-staircase subsections. Also notice that the intersection of the two $(n-1)$-staircases is what is left after both the longest column and the longest row are removed from the n-staircase, which is simply an $(n-2)$-staircase. So far we have counted $2G_{n-1} - G_{n-2}$ rectangular subregions, which includes every rectangular subregion except those subregions that contain square A. It remains only to count the number of subregions that contain square A. Note that in this case one simply needs to choose any other intersection not on the outer edges of the longest column or longest row as the opposite corner to form another rectangular region. The number of choices for such an intersection is a triangular number, since the number of choices starts with one choice in the first column, and the number of choices per column goes up in increasing intervals of one, up to the last column, which has n choices. Thus in fact the number of rectangular regions containing square A is $n(n+1)/2 = \binom{n+1}{2} = \binom{n+1}{n-1}$. Thus G_n is equal to $2 \cdot G_{n-1} - G_{n-2} + \binom{n+1}{2}$. Now, noting that $G_1 = 1 = \binom{4}{4}$ and $G_2 = 5 = \binom{5}{4}$ (by observation), we get $G_3 = 2\binom{5}{4} - \binom{4}{4} + \binom{4}{2} = \binom{5}{4} + \binom{4}{3} + \binom{4}{2} = \binom{5}{4} + \binom{5}{3} = \binom{6}{4}$. If we now inductively assume that $G_k = \binom{k+3}{4}$ for some positive integer k, we get $G_{k+1} = 2G_k - G_{k-1} + \binom{k+2}{2} = 2\binom{k+3}{4} - \binom{k+2}{4} + \binom{k+2}{2} = \binom{k+3}{4} + \binom{k+2}{3} + \binom{k+2}{2} = \binom{k+3}{4} + \binom{k+3}{3} = \binom{k+4}{4} = \binom{(k+1)+3}{4}$ and this completes the inductive proof. Thus we finally get $G_n = \binom{n+3}{4}$ for $n \geq 1$.

The figure shows a 5-staircase.