My Favorite Problems, 6

Harold B. Reiter
University of North Carolina Charlotte

This is the sixth of a series of columns of mathematics problems. I am soliciting future problems for this column from the readers of *M&I Quarterly*. I'm looking for problems with solutions that don't depend on highly technical ideas. Ideal problems should be **easily understood** and accessible to bright high school students. Their solutions should require a clever use of a well-known problem solving technique. Send your problems and solutions by email to me at `hbreiter@email.uncc.edu`. In general, we'll list the problems in one issue and their solutions in the next issue.

6.1 (*South Africa Math Olympiad, 2003*) Fill numbers in the blanks to make each sentence true. Find *all* solutions.
The number of times the digit 0 appears in the puzzle is ...
The number of times the digit 1 appears in the puzzle is ...
The number of times the digit 2 appears in the puzzle is ...
The number of times the digit 3 appears in the puzzle is ...
The number of times the digit 4 appears in the puzzle is ...
The number of times the digit 5 appears in the puzzle is ...
The number of times the digit 6 appears in the puzzle is ...
The number of times the digit 7 appears in the puzzle is ...
The number of times the digit 8 appears in the puzzle is ...
The number of times the digit 9 appears in the puzzle is ....

6.2 A chess king starts at a position A in the top row. The number of paths of length 7 to a position B in the bottom row is a perfect square. The number of paths of length 7 from A to a position C in the bottom row is a perfect cube. The number of paths of length 7 to a position D is both a perfect square and a perfect cube. How many chess king paths of length 5 are there from B to C?

6.3 Find all solutions to

$$\frac{a}{b} + \frac{c}{d} + \frac{e}{f} + \frac{g}{h} = 5i$$

where each letter represents a different nonzero digit.

Problems from My Favorite Problems, 5 with solutions.

5.1 You have three piles of stones containing 5, 49, and 51 stones. You can join any two piles together into one pile and you can divide any pile with an even number of stones into two piles of equal size. Can you ever achieve 105 piles each with one stone?

Solution: Note that both 5 and $49 + 51$ are multiples of 5, so if the first move is to add the two big piles, then every pile after that will have a multiple of 5 stones since half a multiple of 5 is again a multiple of 5. On the other hand, both $5 + 49$ and 51 are multiples of 3, so with this first move, all the piles will have a multiple of 3 stones after that. Finally, $5 + 51 = 56$ which, like 7 is a multiple of 7. No matter what moves are made after that, every pile in sight will have a multiple of 7 stones. Thus, the position of 105 singleton piles can never be achieved.

5.2 You sit at a table that has some coins on it. Each one, of course, shows heads or tails. You are wearing a blindfold and thick gloves, so it is impossible for you to tell by sight or touch what each coin shows. At the outset, you know how many coins are on the table, and how many show heads. You can do whatever you like to the coins, turn them over, etc. as long as all the coins end up back on the table. The question is, how can you divide the coins into two groups so that each has the same number of heads?

Solution: Suppose there are k heads on the table. Take any k coins and make them into a group G. Suppose that among the k coins in G, there are h heads and thus $k-h$ tails. Among the coins not in G there are $k - h$ showing heads. Now turn over all the k coins in G. This produces $k - h$ heads in G. Note that it is not necessary to know how many coins are on the table.