

**My Favorite Problems, 7**  
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This is the seventh of a series of columns of mathematics problems. I am soliciting future problems for this column from the readers of *M&I Quarterly*. I'm looking for problems with solutions that don't depend on highly technical ideas. Ideal problems should be **easily understood** and accessible to bright high school students. Their solutions should require a clever use of a well-known problem solving technique. Send your problems and solutions by email to me at [hbreiter@email.uncc.edu](mailto:hbreiter@email.uncc.edu). In general, we'll list the problems in one issue and their solutions in the next issue.

7.1 Two integers are called *approximately equal* if their difference is at most 1. How many different ways are there to write 2005 as a sum of one or more positive integers which are all approximately equal to each other? The order of terms does not matter: two ways which only differ in the order of terms are not considered different.

7.2 Find all ordered pairs of  $(x, y)$  of positive integers that satisfy

$$x^3 + y^3 = (x + y)^2.$$

7.3 Call a rational number  $s = a/b$  in the interval open  $(0, 1)$  a *skipable number* if there is a sequence  $a_n$  of 0's and 1's such that if  $r_n = \sum a_n/n$ , that is, the ratio of the number of ones to entries, then  $r_n < s$  for some  $n$  and  $r_n > s$  for a larger  $n$ , but  $r_n = s$  for no  $n$ . Find all skipable numbers?

Problems from My Favorite Problems, 6 with solutions.

6.1 Find all solutions to

$$\frac{a}{b} + \frac{c}{d} + \frac{e}{f} + \frac{g}{h} = 5i$$

where each letter represents a different nonzero digit.

**Solution:** First note that  $i = 1, 2$  or  $3$  since the largest possible value of  $\frac{a}{b} + \frac{c}{d} + \frac{e}{f} + \frac{g}{h}$  is  $9/1 + 8/2 + 7/3 + 4/6 = 16$ . When  $i = 1$  there are only four solutions to  $\frac{a}{b} + \frac{c}{d} + \frac{e}{f} + \frac{g}{h} = k$ , and the values of  $k$  are 3, 6, and 8, not 5. When  $i = 3$ , there are just five solutions to  $\frac{a}{b} + \frac{c}{d} + \frac{e}{f} + \frac{g}{h} = k$ , and the values of  $k$  are 10, 12, 13 and 14, not 15. However, when  $i = 2$  there are three solutions to  $\frac{a}{b} + \frac{c}{d} + \frac{e}{f} + \frac{g}{h} = k$ , and the values of  $k$  are 7, 12, and 10. The solution we seek is  $5/1 + 7/3 + 8/4 + 6/9 = 5 \cdot 2 = 10$ .

6.2 (*South Africa Math Olympiad, 2003*) Fill numbers in the blanks to make each sentence true. Find *all* solutions.

The number of times the digit 0 appears in the puzzle is ...

The number of times the digit 1 appears in the puzzle is ...

The number of times the digit 2 appears in the puzzle is ...

The number of times the digit 3 appears in the puzzle is ...

The number of times the digit 4 appears in the puzzle is ...

The number of times the digit 5 appears in the puzzle is ...

The number of times the digit 6 appears in the puzzle is ...

The number of times the digit 7 appears in the puzzle is ...

The number of times the digit 8 appears in the puzzle is ...

The number of times the digit 9 appears in the puzzle is ....

**Solution:** There are two solutions, one with the sum of the entered numbers 20 and the other with sum 21.

digit	0	1	2	3	4	5	6	7	8	9
A	1	7	3	2	1	1	1	2	1	1
B	1	11	2	1	1	1	1	1	1	1

Call the solutions A and B. We'll argue that there is only one solution for which the sum of the numbers in row two is 20 and only one for which the sum is 21. Note first that every digit appears at least once, so the digit 0 must appear exactly once. This means that the first entry in each solution must be 1. Since we have two rows and there are at least 10 digits in each row, there are at least 20 digits altogether. We first argue that there are either 20 or 21 digits. If there were 22 or more digits, this would mean that there are two two-digit numbers in the second row or a three-digit number in the second row. Clearly neither of these can happen. Since the number in the second row tells us how many digits of each type there are in the two rows, the sum of the entries in the second row is either 20 or 21.

Case A. Suppose the sum is 20 and there are just two 1's. Then all the rest of the entries in columns 2 through 9 are at least 2. In order that the sum be 20, the number of 2's among these entries must be at least 8, meaning the number of 2's is at least 8, which makes the sum too big. Trying three 1's leads to a similar contradiction, as does four, five and six. When we get to seven, we see that the number of 7's would have to be 2, and the seven 1's could be in the locations 0, 4, 5, 6, 8, and 9 in addition to the one 1 in the first row. Putting a 2 in the line opposite 7 requires that we put at least a 2 opposite 2. But putting a 2 in that spot means there are three 2's, which means the 2 is wrong. Try a 3 in that spot and see that this requires a 2 in the entry opposite 3, which is just what we need.

Case B. There must be exactly one two digit number, and it must be the number 11. It cannot be 10 because that would mean there were two 0's, hence at least two 2's. But that 10 must go opposite the number 1, and there cannot be enough 1's in the two rows to total 10, given that the entries opposite 0 and 2 are not 1's. So let us try to find eleven 1's in the two rows. If we agree that we will not include any of the digits 3, 4, 5, 6, 7, 8 and 9 more than once, we get: one 1 in the first row, two 1's opposite 1, and seven more ones in the others positions for a total of eleven 1's. But we have not assigned a number to 2. Clearly a 2 works.

6.3 A chess king starts at a position A in the top row. The number of paths of length 7 from A to a position B in the bottom row is a perfect square. The number of paths of length 7 from A to a position C in the bottom row is a perfect cube. The number of paths of length 7 to a position D is both a perfect square and a perfect cube. How many chess king paths of length 5 are there from B to C?

A							
	196			27		1	

**Solution:**

Position A is the upper left corner (or right corner), B is the second square and C is the sixth square of the bottom row. To count the paths of length 5 from B to C, build five arrays,  $A_1, A_2, \dots, A_5$  with the number of paths of length  $i$  that end at each square. The answer is 60.