My Favorite Problems, 8
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This is the eighth of a series of columns of mathematics problems. I am soliciting future problems for this column from the readers of *M&I Quarterly*. I'm looking for problems with solutions that don't depend on highly technical ideas. Ideal problems should be **easily understood** and accessible to bright high school students. Their solutions should require a clever use of a well-known problem solving technique. Send your problems and solutions by email to me at `hbreiter@email.uncc.edu`. In general, we'll list the problems in one issue and their solutions in the next issue.

8.1 Submitted by Brian Smith. Which of the following 5 sentences are true?

(a) It is not the case that 2 consecutive sentences are both false.

(b) There are fewer false than true sentences.

(c) It is not the case that 3 consecutive sentences are all false.

(d) It is not the case that 2 consecutive sentences are both true.

(e) There are exactly 3 false sentences.

8.2 A collection (also called a multiset) of positive integers (not necessarily distinct) is called Kool if the sum of all its elements equals their product. For example, $\{2, 2, 2, 1, 1\}$ is a Kool multiset. This problem comes from the Stetson University mathematics contest. See `http://www.stetson.edu/departments/mathcs/events/mathcontest/index.shtml`.

(a) Show that there exists a Kool multiset of n numbers for all $n > 1$.

(b) Find all Kool multisets with sums of 100.

(c) Find all Kool multisets with 100 members.

8.3 [College Math Journal, March 2005] Tom picks a fourth degree polynomial p with nonnegative integer coefficients. Sally claims that she can ask Tom just two values of p and then tell him the five coefficients. She chooses a and he tells her $p(a)$. Then she chooses b and he tells her $p(b)$. What is the winning strategy? For example, Sally asks for $p(1)$ and gets the answer 9. Then she asks for $p(11)$ and gets the answer 46,709. What is the polynomial, and how did she know it?

Problems from My Favorite Problems, 7, with solutions.

7.1 Two integers are called *approximately equal* if their difference is at most 1. How many different ways are there to write 2005 as a sum of one or more positive integers which are all approximately equal to each other? The order of terms does not matter: two ways which only differ in the order of terms are not considered different.

Solution: 2005. Let $G(n)$ denote the number of ways to write n as a sum of approximately equal positive integers. Trial and error produces $G(1) = 1, G(2) = 2, G(3) = 3$ and $G(4) = 4$. In fact, we can prove by mathematical induction that $G(n) = n$ for all positive integers. Suppose $n = a_1 + a_2 + \cdots a_k$ where either all the a_i are the same or there are two different values and they differ by 1. Thus we have either $n = ka + m(a-1), m > 0, k > 0$ or $n = ka, k > 0$. In the first case $n + 1 = (k+1)a + (m-1)(a-1)$ if $m > 0$ and, in the second case $n + 1 = a + 1 + (k-1)a$. In addition $n = 1 + 1 + 1 + \cdots + 1$ represents a new representation. Thus $G(n+1) = G(n) + 1$ for all integers n.

7.2 Find all ordered pairs of (x, y) of positive integers that satisfy

$$x^3 + y^3 = (x+y)^2.$$

Solution: Factor $x^3 + y^3$ and reduce the equation to $x^2 - xy + y^2 = x + y$ by removing a factor of $x + y$ from both sides. Now use the quadratic formula on $x^2 - xy - x + y^2 - y = 0$ to get

$$x = \frac{y + 1 \pm \sqrt{(y+1)^2 - 4y(y-1)}}{2} = \frac{y + 1 \pm \sqrt{-3y^2 + 6y + 1}}{2}.$$

Let $g(y) = -3y^2 + 6y + 1$. The the only positive integer values for which $g(y) > 0$ are $y = 1, 2$ and $g(1) = 4, g(2) = 1$. These lead to the only three solutions $x = \frac{1+1\pm\sqrt{g(1)}}{2} = 2$, and $x = \frac{2+1\pm\sqrt{g(2)}}{2} = 1, 2$, so there are exactly three solutions, $(1, 2), (2, 1)$ and $(2, 2)$.

7.3 Call a rational number $s = a/b$ in the interval $(0, 1)$ a 'skipable number' if there is a sequence a_n of 0s and 1s such that if $r_n = \sum a_n/n$, that is, the ratio of the number of ones to entries, then $r_n < s$ for some n and $r_n > s$ for a larger n, but $r_n = s$ for no n. Find all skipable numbers?

Solution: This solution is due to James Rudzinski. The only numbers which are not skipable are numbers of the form $k/(k+1)$ for any $k \geq 1$. If a rational number a/b, where a and b are integers, can be skipped, then there must exist a rational number x/y, x and y integers, in the sequence r_n such that the next number in the sequence actually skips the number. Then we have $x/y < a/b$, and $(x+1)/(y+1) > a/b$ for some positive integers x and y. Then we get $bx < ay,\ bx + b > ay + a$ so $bx + b - a > ay$ and it follows that $bx < ay < bx + b - a$. If $b - a = 1$, there can be no x and y which satisfy the property, that is, the number a/b cannot be skipped. Finally, the sequence a_n given by $0, 1, 1, 1, 1, 1, \ldots$ generates the sequence $0, 1/2, 2/3, 3/4, 4/5, \ldots$ which clearly skips all numbers not of the form $k/(k+1)$, so these are the only non-skipable numbers. Also, if you changed the definition of skipable so that you wanted a larger fraction first and then later a smaller fraction while skipping the number, then by a very similar argument you get the only non skipable numbers are of the form $1/k, k > 1$, and the sequence $1, 0, 0, 0, 0, \ldots$ skips all numbers but those. Finally if you extended the definition of skipable to mean potentially from either direction then clearly $1/2$ is the only non-skipable number.