My Favorite Problems, 9

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This is the ninth of a series of columns of mathematics problems. I am soliciting future problems for this column from the readers of *M&I Quarterly*. I'm looking for problems with solutions that don't depend on highly technical ideas. Ideal problems should be **easily understood** and accessible to bright high school students. Their solutions should require a clever use of a well-known problem solving technique. Send your problems and solutions by email to me at `hbreiter@email.uncc.edu`. In general, we'll list the problems in one issue and their solutions in the next issue.

9.1 The Infected Checkerboard. This problem appears in the essay Five Algorithmic Puzzles by Peter Winkler in A Tribute to a Mathemagician. He attributed the problem to KVANT around 1986. Given a 10×10 checkerboard of squares, some of which are darkened (infected). The infection spreads among the squares as follows: if a square has two or more infected neighbors, then it becomes infected. Neighboring means sharing an edge. If we start with all the squares on the main diagonal infected, clearly all the squares will eventually become infected. Prove that no fewer than 10 squares can accomplish the complete contamination of the board.

9.2 Two perfect logicians, S and P, are told that integers x and y have been chosen such that $1 < x < y$ and $x + y < 100$. S is given the value $x + y$ and P is given the value xy. They then have the following conversation. P: I cannot determine the two numbers. S: I knew that. P: Now I can determine them. S: So can I. Given that the above statements are true, what are the two numbers? This is problem 3 at Nick Hobson's website Nick's Mathematical Puzzles `http://www.qbyte.org/puzzles/p003s.html` Also, see `http://www.mathematik.uni-bielefeld.de/~sillke/PUZZLES/logic_sum_product`

9.3 Find all integers $1 \leq k \leq 169$ for which 169 is the sum of k nonzero squares. The squares are not necessarily unique. For example $k = 5$: $169 = 1 + 4 + 4 + 16 + 144$.

Problems from My Favorite Problems, 8, with solutions.

8.1 Which of the following 5 sentences are true? Submitted by Brian Smith.

(a) It is not the case that 2 consecutive sentences are both false.

(b) There are fewer false than true sentences.

(c) It is not the case that 3 consecutive sentences are all false.

(d) It is not the case that 2 consecutive sentences are both true.

(e) There are exactly 3 false sentences.

Solution: The answer is $ffftf$. We are indebted to Steve Coward, who pointed out the flaw in the original 'solution' and supplied a solution. Let t denote the number of sentences that are true. Then $0 \leq t \leq 5$. We show that $t = 1$ and then find that precisely statement (c) is true.

$t = 0$. This contradicts (d).

$t = 2$. Then (b) is false. There cannot be three consecutive false statements, so (c) is true. Now if (d) is true, then there are two consecutive true sentences, which makes (d) false. On the other hand, if (d) is false, then there cannot be two consecutive true statements, so (d) is true. Thus, we have a contradiction.

$t = 3$. In this case, (b) is true because there are two false statements and three true ones. Also, (e) is false and (c) is true. Now if (a) is true, then (d) is false, but this implies that there are two consecutive false statements, which contradicts (a). On the other hand, if (a) is false, the (d) must be true in order to have $t = 3$, but in this case, there are two consecutive true statements contradicting (d).

$t = 4$. In this case, clearly both (d) and (e) are false, a contradiction.

$t = 5$. Again both (d) and (e) are false.

This leaves just the case $t = 1$. Now (d) is true. And we can make (c) false by making (c) false. The string $ffftf$ makes (a) also false. Of course, (b) is false as well.

8.2 A collection (also called a multiset) of positive integers (not necessarily distinct) is called Kool if the sum of all its elements equals their product. For example, $\{2, 2, 2, 1, 1\}$ is a Kool multiset. This problem comes from the Stetson University mathematics contest. See `http://www.stetson.edu/departments/mathcs/events/mathcontest/index.shtml`.

(a) Show that there exists a Kool multiset of n numbers for all $n > 1$.

(b) Find all Kool multisets with sums of 100.

(c) Find all Kool multisets with 100 members.

Solution:

(a) The multiset $T = \{2, b, 1, 1, 1 \ldots, 1\}$ with $b - 2$ 1's is a Kool multiset with $2 + (b - 2) = b$ members. We can write this as $\{2, b, 1^{b-2}\}$

(b) There are nine. Each factorization of 100 into integers larger than 1 gives rise to a Kool multiset with sum 100. For example, $100 = 4 \times 5 \times 5$ and $\{4, 5, 5, 1^{100-4-5-5}\}$ is a Kool multiset. There are nine such factorizations.

(c) Condition on the number of summands larger than 1. In case there are two such summands, a and b, there are three Kool multisets. In fact, if the collection is of the form $\{a, b, 1^{98}\}$, then we must have $ab - a - b + 1 = 98 + 1$, or $(a-1)(a-1) = 99$. This has the three solutions $(a, b) = (100, 2), (34, 4), (12, 10)$, which lead to the first three Kool multisets below. Trial and error for the case of fewer 1's gives the other two Kool multisets with 100 numbers: $\{100, 2, 1^{98}\}, \{34, 4, 1^{98}\}, \{12, 10, 1^{98}\}, \{7, 42, 1^{97}\}, \{33, 22, 1^{95}\}$.

8.3 [College Math Journal, March 2005] Tom picks a fourth degree polynomial p with nonnegative integer coefficients. Sally claims that she can ask Tom just two values of p and then tell him the five coefficients. She chooses a and he tells her $p(a)$. Then she chooses b and he tells her $p(b)$. What is the winning strategy? For example, Sally asks for $p(1)$ and gets the answer 9. Then she asks for $p(11)$ and gets the answer 46,599. What is the polynomial, and how did she know it?

Solution: Sally asks Tom for $p(1)$. His answer is the sum of the coefficients of p. She knows that $p(1)$ is at least as big as any of p's coefficients. She takes any number b larger than $p(1)$ and asks for $p(b)$. Then she computes the base b representation of $p(b)$. This uniquely describes the coefficients of p. In the example, she could have asked for $p(10)$ and gotten the answer 32,013. She would then have known that the polynomial was $3x^4 + 2x^3 + x + 3$. Asking for $p(11)$ disguises the strategy a bit better. Of course, the base 11 representation of 46,599 is 32013_{11}.