UNC Charlotte 2005 Comprehensive

March 7, 2005

1. The numbers x and y satisfy $2^x = 15$ and $15^y = 32$. What is the value xy?

   (A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** none of **A**, **B**, **C** or **D**

2. Suppose x, y, z, and w are real numbers satisfying $x/y = 4/7$, $y/z = 14/3$, and $z/w = 3/11$. When $(x+y+z)/w$ is written in the form m/n where m and n are positive integers with no common divisors bigger than 1, what is $m+n$?

   (A) 20 **(B)** 26 **(C)** 32 **(D)** 36 **(E)** 37

3. Let m be an integer such that $1 \leq m \leq 1000$. Find the probability of selecting at random an integer m such that the quadratic equation

$$6x^2 - 5mx + m^2 = 0$$

   has at least one integer solution.

   (A) 0.333 **(B)** 0.5 **(C)** 0.667 **(D)** 0.778 **(E)** 0.883

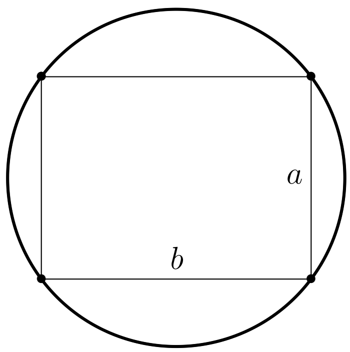
4. Let $z = x + iy$ be a complex number, where x and y are real numbers. Let A and B be the sets defined by

$$A\{z \mid |z| \leq 2\} \text{ and } B = \{z| \ (1-i)z + (1+i)\bar{z} \geq 4\}.$$

   Recall that $\bar{z} = x - iy$ is the conjugate of z and that $|z| = \sqrt{x^2 + y^2}$. Find the area of the region $A \cap B$.

   (A) $\pi/4$ **(B)** $\pi - 2$ **(C)** $(\pi - 2)/4$ **(D)** 4π **(E)** $\pi - 4$

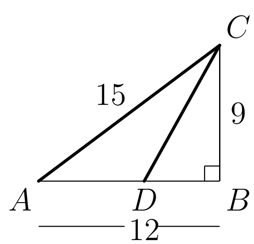
5. The area of the $a \times b$ rectangle shown in the picture below is $\frac{6}{5\pi}$ of the area of the circle. Assuming $b > a$, what is the value of $\frac{b}{a}$?



(A) 2 **(B)** 2.5 **(C)** 3 **(D)** 3.25 **(E)** 3.5

6. In the right triangle ABC the segment CD bisects angle C, $AC = 15$, and $BC = 9$. Find the length of $\overline{CD}$.

(A) $9\sqrt{5}/2$ **(B)** 11 **(C)** $9\sqrt{6}/2$ **(D)** 12 **(E)** $15\sqrt{3}/2$



7. The product of the two roots of the equation $\log x + \log(x+2) = 3$ is equal to

(A) $-\log 2$ **(B)** -10^3 **(C)** $\log 2$ **(D)** 10^3 **(E)** The equation has only one root

8. The polynomial $p(x) = 2x^4 - x^3 - 7x^2 + ax + b$ is divisible by $x^2 - 2x - 3$ for certain values of a and b. What is the sum of a and b?

(A) -34 **(B)** -30 **(C)** -26 **(D)** -18 **(E)** 30

9. One hundred monkeys have 100 apples to divide. Each adult gets three apples while three children share one. How many adult monkeys are there?

   (A) 10 **(B)** 20 **(C)** 25 **(D)** 30 **(E)** 33

10. Let x and y be the positive integer solution to the equation

$$\frac{1}{x+1} + \frac{1}{y-1} = 5/6.$$

   Find $x + y$.

   (A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

11. Let x and y be two integers that satisfy all of the following properties:

   (a) $5 < x < y$,

   (b) x is a power of a prime and y is a power of a prime, and

   (c) the quantities $xy + 3$ and $xy - 3$ are both primes.

   Among all the solutions, let (x, y) be the one with the smallest product. Which of the following statements is true? Note the list of primes on the last page.

   (A) $x + y$ is a perfect square **(B)** the number xy is prime

   (C) $y = x + 3$ **(D)** $y = x + 1$ **(E)** $x + y = 17$

12. In a 10-team baseball league, each team plays each of the other teams 18 times. No game ends in a tie, and, at the end of the season, each team is the same positive number of games ahead of the next best team. What is the greatest number of games that the last place team could have won.

   (A) 27 **(B)** 36 **(C)** 54 **(D)** 72 **(E)** 90

13. Suppose f is a real function satisfying $f(x + f(x)) = 4f(x)$ and $f(1) = 4$. What is $f(21)$?

   (A) 16 **(B)** 21 **(C)** 64 **(D)** 105 **(E)** none of **A**, **B**, **C** or **D**

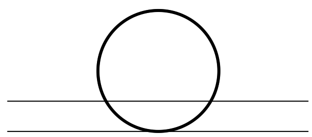
14. Suppose $\sin\theta + \cos\theta = 0.8$. What is the value of $\sin(2\theta)$?

   (A) -0.36 **(B)** -0.16 **(C)** 0 **(D)** 0.16 **(E)** 0.36

15. The product of five consecutive positive integers divided by the sum of the five integers is a multiple of 100. What is the least possible sum of the five integers?

    (A) 605 **(B)** 615 **(C)** 620 **(D)** 625 **(E)** 645

16. Two parallel lines are one unit apart. A circle of radius 2 touches one of the lines and cuts the other line. The area of the circular cap between the two parallel lines can be written in the form $a\pi/3 - b\sqrt{3}$. Find the sum $a + b$ of the two integers a and b.

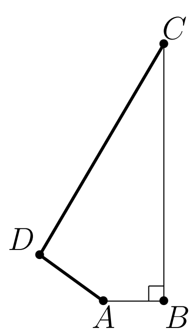


    (A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

17. For each positive integer N, define $S(N)$ as the sum of the digits of N and $P(N)$ as the product of the digits. For example, $S(1234) = 10$ and $P(1234) = 24$. How many four digit numbers N satisfy $S(N) = P(N)$?

    (A) 0 **(B)** 6 **(C)** 12 **(D)** 24 **(E)** None of the above

18. Consider a quadrilateral $ABCD$ with $AB = 4$, $BC = 10\sqrt{3}$, and $\angle DAB = 150°$, $\angle ABC = 90°$, and $\angle BCD = 30°$. Find DC.



    (A) 16 **(B)** 17 **(C)** 18 **(D)** 19 **(E)** 20

19. Solve the equation $\frac{\sqrt{x+1}+\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}} = 3$ for x.

    (A) 4/3 **(B)** 5/3 **(C)** 7/5 **(D)** 9/5 **(E)** none of **A**, **B**, **C** or **D**

20. Two rational numbers r and s are given. The numbers $r+s, r-s, rs$, and s/r are computed and arranged in order by value to get the list $1/3, 3/4, 4/3, 7/3$. What is the sum of the squares of r and s?

    (A) 9/25 **(B)** 4/9 **(C)** 9/4 **(D)** 25/9 **(E)** 6

21. How many pairs of positive integers satisfy the equation $3x + 6y = 95$?

    (A) none **(B)** one **(C)** two **(D)** three **(E)** four

22. Suppose all three of the points $(-2, 10), (1, -8)$, and $(4, 10)$ lie on the graph of $y = ax^2 + bx + c$. What is abc?

    (A) -24 **(B)** 0 **(C)** 12 **(D)** 24 **(E)** 48

23. An amount of \$2000 is invested at $r\%$ interest compounded continuously. After four years, the account has grown to \$2800. Assuming that it continues to grow at this rate for 16 more years, how much will be in the account?

    (A) \$8976.47 **(B)** \$9874.23 **(C)** \$10001.99

    (D) \$10756.48 **(E)** \$2004.35

24. Two sides of an isosceles triangle have length 5 and the third has length 6. What is the radius of the inscribed circle?

    (A) 1 **(B)** 1.25 **(C)** 1.5 **(D)** 1.75 **(E)** 2

25. Consider the circle $x^2 - 14x + y^2 - 4y = -49$. Let L_1 and L_2 be lines through the origin O that are tangent to the circle at points A and B. Which of the following is closest to the measure of the angle AOB?

    (A) 35.1° **(B)** 34.8° **(C)** 33.6° **(D)** 32.8° **(E)** 31.9°

26. Three digits a, b, and c are selected from the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, one at a time, with repetition allowed. What is the probability that $a > b > c$?

    (A) 3/25 **(B)** 4/25 **(C)** 1/5 **(D)** 6/25 **(E)** 7/25

27. Let p denote the smallest prime number greater than 200 for which there are positive integers a and b satisfying

$$a^2 + b^2 = p.$$

What is $a + b$? Note the list of primes on the last page.

(A) 16 **(B)** 17 **(C)** 18 **(D)** 19 **(E)** 20

28. Four positive integers a, b, c and d satisfy $abcd = 10!$. What is the smallest possible sum $a + b + c + d$?

(A) 170 **(B)** 175 **(C)** 178 **(D)** 183 **(E)** 185

29. The faces of a cube are colored red and blue, one at a time, with equal probability. What is the probability that the resulting cube has at least one vertex P such that all three faces containing P are colored red?

(A) 1/4 **(B)** 5/16 **(C)** 27/64 **(D)** 1/2 **(E)** None of the above

30. Let a, b, c, and d denote four digits, not all of which are the same, and suppose that $a \le b \le c \le d$. Let n denote any four digit integer that can be built using these digits. Define $K(n) = \underline{dcba} - \underline{abcd}$. The function K is called the *Kaprekar* function. For example $K(1243) = 4321 - 1234 = 3087$. A four-digit integer M is called a Kaprekar number if there is a four-digit integer N such that $K(N) = M$. Which of the following is not a Kaprekar number?

(A) 2936 **(B)** 7263 **(C)** 5265 **(D)** 3996 **(E)** 6264

List of Primes between 1 and 500:

2	3	5	7	11	13	17	19	23	29
31	37	41	43	47	53	59	61	67	71
73	79	83	89	97	101	103	107	109	113
127	131	137	139	149	151	157	163	167	173
179	181	191	193	197	199	211	223	227	229
233	239	241	251	257	263	269	271	277	281
283	293	307	311	313	317	331	337	347	349
353	359	367	373	379	383	389	397	401	409
419	421	431	433	439	443	449	457	461	463
467	479	487	491	499					