UNC Charlotte 2006 Comprehensive

March 6, 2006

1. What is the sum of the digits of the integer solution to $\sqrt{14+\sqrt{27-\sqrt{x-1}}}=4$?

   (A) 5 **(B)** 6 **(C)** 8 **(D)** 9 **(E)** 11

2. In a box there are red and blue balls. If you select a handful of them with eyes closed, you have to grab at least 5 of them to make sure at least one of them is red and you have to grab at least 10 of them to make sure both colors appear among the balls selected. How many balls are there in the box?

   (A) 10 **(B)** 11 **(C)** 12 **(D)** 13 **(E)** 14

3. Some hikers start on a walk at 9 a.m. and return at 2 p.m. One quarter of the distance walked is uphill, one half is level, and one quarter is downhill. If their speed is 4 miles per hour on level land, 2 miles per hour uphill, and 6 miles per hour downhill, approximately how far did they walk?

   (A) 16.4 miles **(B)** 17.1 miles **(C)** 18.9 miles

   (D) 20.0 miles **(E)** 21.2 miles

4. Five regular polygons, a triangle, a square, a pentagon, a hexagon, and a dodecagon (a 12-sided polygon), all have the same perimeter. Which one has the greatest area?

   (A) the triangle **(B)** the square **(C)** the pentagon

   (D) the hexagon **(E)** the dodecagon

5. The radius of the circle given by

$$x^2 - 6x + y^2 + 4y = 12$$

   is

   (A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 36

6. What is the length of the interval of solutions to the inequality $1 \leq 3-4x \leq 11$?

   (A) 1.75 **(B)** 2.00 **(C)** 2.25 **(D)** 2.50 **(E)** 3.25

7. Quadrilateral $ABCD$ with the sides $AB = 20, BC = 7, CD = 24$ and $DA = 15$ has right angles at A and C. What is the area of $ABCD$?

   (A) 154 **(B)** 186 **(C)** 200 **(D)** 234 **(E)** 286

8. Benny eats a box of cereal in 14 days. He eats the same size box of cereal with his younger brother Nathan in 10 days. How many days will it take Nathan to finish the box of cereal alone?

   (A) 20 **(B)** 25 **(C)** 30 **(D)** 35 **(E)** 40

9. The base of a regular square pyramid is inscribed in the base of a cylinder. The height of the cylinder is triple the height of the pyramid. Find the ratio of the volume of the pyramid to the volume of the cylinder.

   (A) $\frac{2}{9\pi}$ **(B)** $\frac{2}{3\pi}$ **(C)** $\frac{\pi}{9}$ **(D)** $\frac{4\pi}{3}$ **(E)** $\frac{4\pi}{9}$

10. Given the following system of equations

$$\begin{aligned} \frac{1}{x}+\frac{1}{y} &= \frac{1}{3} \\ \frac{1}{x}+\frac{1}{z} &= \frac{1}{5} \\ \frac{1}{y}+\frac{1}{z} &= \frac{1}{7} \end{aligned}$$

What is the value of the ratio $\frac{z}{y}$?

(A) 17 **(B)** 23 **(C)** 29 **(D)** 31 **(E)** 36

11. Suppose a, b, c are integers such that

1. $0 < a < b$,

2. The polynomial $x(x-a)(x-b)-17$ is divisible by $(x-c)$.

What is $a+b+c$?

(A) 14 **(B)** 17 **(C)** 21 **(D)** 24 **(E)** 27

12. Let x, y be positive integers with $x > y$. If $1/(x+y)+1/(x-y) = 1/3$, find x^2+y^2.

(A) 52 **(B)** 58 **(C)** 65 **(D)** 73 **(E)** 80

13. A ball is dropped onto a floor from a height of 1 meter. Each time that the ball hits the floor it rebounds to half its previous height. (After falling one meter it rebounds to a height of 1/2 meter. The next time it hits the floor, it rebounds to a height of 1/4 meter, etc.). How far has the ball traveled when it hits the floor for the 40th time?

(A) $T = 2+(2^{38}-1)/2^{38}$ **(B)** $T = 1+(2^{38}-1)/2^{39}$ **(C)** $T = 2$

(D) $T = 3$ **(E)** $T > 3$

14. Let a and b be two positive integers such that b is a multiple of a. If $\log_{10}(b/a)^{b/2} + \log_{10}\left(\sqrt{a/b}\right)^{9a} = 1$, then $b^2-a^2 =$

(A) 357 **(B)** 396 **(C)** 1600 **(D)** 5967 **(E)** 8436

15. A triangle with sides $a = 15$, $b = 28$ and $c = 41$ has an altitude of integer length. What is the length of this altitude?

(A) 6 **(B)** 7 **(C)** 9 **(D)** 16 **(E)** 17

16. Inside the unit circle $D = \{(x, y) \mid x^2 + y^2 = 1\}$ there are three smaller circles of equal radius a, tangent to each other and to D. If $a = p\sqrt{3} - q$, find the sum of the integers $p + q$.

(A) 4 **(B)** 5 **(C)** 7 **(D)** 10 **(E)** 12

17. Let N denote a six-digit integer whose 6 digits are 1,2,3,4,5,6 in random order. What is the probability that N is divisible by 6?

(A) 1/6 **(B)** 1/3 **(C)** 2/5 **(D)** 1/2 **(E)** 3/5

18. Three fair dice are rolled. What is the probability that the product of the three outcomes is a prime number? Recall that 1 is not considered to be prime.

(A) 0 **(B)** 1/72 **(C)** 1/36 **(D)** 1/24 **(E)** 1/8

19. The odd numbers from 1 to 17 can be used to build a 3×3 magic square (the rows and columns have the same sum). If the $1, 5$, and 13 are as shown, what is x?

(A) 7 **(B)** 9 **(C)** 11 **(D)** 15 **(E)** 17

	1	
5		13
x		

20. Let N denote the two-digit number whose cube root is the square root of the sum of its digits. How many positive divisors does N have?

(A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

21. At a picnic there were c children, f adult females, and m adult males, where $2 \leq c < f < m$. Every person shook hands with every other person. The sum of the number of handshakes between children, the number of handshakes between adult females and the number of handshakes between adult males is 57. How many handshakes were there altogether?

    (A) 153 **(B)** 171 **(C)** 190 **(D)** 210 **(E)** 231

22. Two consecutive positive integers n and $n+1$, both with exactly four divisors, have the same sum of divisors. What is the number of divisors of their product?

    (A) 6 **(B)** 12 **(C)** 16 **(D)** 20 **(E)** 24

23. Suppose a, b, c, and d are positive integers satisfying

$$ab + cd = 38$$

$$ac + bd = 34$$

$$ad + bc = 43$$

    What is $a + b + c + d$?

    (A) 15 **(B)** 16 **(C)** 17 **(D)** 18 **(E)** 20

24. Let $a_0 = 2$ and $a_1 = 3$ and let $a_{n+2} = |a_n| - a_{n+1}$ for all $n \geq 0$. What is the smallest n such that $a_n \geq 100$?

    (A) 10 **(B)** 11 **(C)** 13 **(D)** 15 **(E)** 17

25. Let $S(n) = n$ in case n is a single digit integer. If $n \geq 10$ is an integer, $S(n)$ is the sum of the digits of n. Let N denote the smallest positive integer such that $N + S(N) + S(S(N)) = 99$. What is $S(N)$?

    (A) 9 **(B)** 10 **(C)** 12 **(D)** 15 **(E)** 18

26. Find the number of odd divisors of 7!.

    (A) 4 **(B)** 6 **(C)** 10 **(D)** 12 **(E)** 24

27. John was contracted to work A days. For each of these A days that John actually worked, he received B dollars. For each of these A days that John didn't work, he had to pay a penalty of C dollars. After the A days of contracted work was over, John received a net amount of D dollars for his work. How many of the A days of contracted work did John not work?

(A) $(AB - D)/(B + C)$ **(B)** $(AB + D)/(B + C)$ **(C)** $(AB - D)/(B - C)$
(D) $(AB + D)/(B - C)$ **(E)** $(AC - B)/(D - C)$

28. Let Γ be a plane containing three points $A(1, 0, 0), B(0, 2, 0), C(0, 0, 1)$. Find the distance from the origin $(0, 0, 0)$ to the plane Γ.

(A) $1/3$ **(B)** $2/3$ **(C)** 1 **(D)** $\pi/3$ **(E)** $2\pi/3$

29. Let N denote the 180-digit number obtained by listing the 90 two-digit numbers from 10 to 99 in order. Thus $N = 10111213\ldots99$. What is the remainder when N is divided by 99?

(A) 0 **(B)** 10 **(C)** 45 **(D)** 54 **(E)** 90

30. What is the length of the shortest path $APQB$ in the plane, where $A = (2, 3)$, $B = (5, 1)$, P lies on the y-axis, and Q lies on the x-axis.

(A) 7 **(B)** 8 **(C)** $\sqrt{65}$ **(D)** $5\sqrt{3}$ **(E)** 9