Test with Solutions for Sponsors

1. The lengths of all three sides of a triangle have integer values and are all different. The area of this triangle is positive. The largest of the lengths equals 4. Find the smallest length of the sides of this triangle.

   (A) 1 (B) 2 (C) 3 (D) 4 (E) None of the above

   Answer: B.
   Solution. The correct answer is B. If the lengths are denoted by x, y, z and $z = 4$, then one has to realize that there is only one point of the plane with integer coordinates x and y situated in the region $x + y > 4$, $0 \leqslant x < 4$, $x < y$, $0 \leqslant y < 4$.

2. Let x and y be two positive integers such that $1 \leqslant x < y \leqslant 9$. Let a be the number with the decimal expansion xy and let b be the number with the decimal expansion yx. Assume that $a + b = 110$. For how many different x does such a y exist?

   (A) 1 (B) 2 (C) 3 (D) 4 (E) 0

   Answer: D.
   Solution. Indeed, the only additional condition on x and y is the restriction $x + y = 10$.

3. A child has 100 plastic figurines, some of which are 4-legged dinosaurs and some of which are 2-legged dinosaurs. If there is a total of 260 legs, how many 4-legged dinosaurs are in the collection?

   (A) 30 (B) 50 (C) 65 (D) 70 (E) 85

   Answer: A.
   Solution. Suppose first that all the dinosaurs are two-legged. Then there are only 200 legs in total, so some dinosaurs should be replaced with four-legged ones. Each such replacement increases the total number of legs by 2, so we need $60/2 = 30$ replacements. Hence $n = 30$.

4. Suppose that x satisfies the equation $\sin(x) = 1/\tan(x)$. Compute $\cos(x)$.

   (A) $\dfrac{-1-\sqrt{5}}{2}$ (B) $\dfrac{\sqrt{3}}{2}$ (C) $\dfrac{\sqrt{5}-1}{2}$ (D) $\dfrac{\sqrt{5}}{4}$ (E) 1

   Answer: C.
   Solution. Multiplying by $\sin(x)$ and simplifying yields $\sin^2(x) = \cos(x)$. Let $u = \cos(x)$. Then, since $\sin^2(x) + \cos^2(x) = 1$, $u = 1 - u^2$. Applying the quadratic formula and throwing out the root $u = (-\sqrt{5} - 1)/2$ (since $|u| \leqslant 1$) yields $u = (\sqrt{5} - 1)/2$.

5. Tom Sawyer and Huck Finn want to paint a fence. Tom can paint the fence by himself in 3 hours, and Huck can paint the fence by himself in 4 hours. At 12 : 00 noon they start painting the fence together. However, at some point they get into a fight. They fight for 10 minutes, during which time no painting gets done. After the fight, Huck leaves and Tom finishes painting alone. If Tom finishes painting at 2 : 25 pm, at what time did the fight begin?

   (A) 12 : 30 (B) 1 : 00 (C) 1 : 10 (D) 1 : 15 (E) 1 : 30

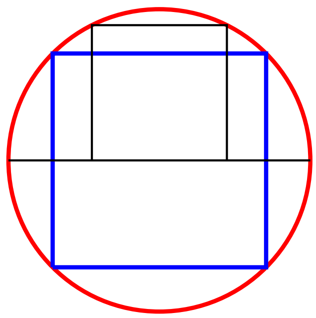
   Answer: B.
   Solution. If they start fighting at t pm, then at that time they have completed $t/3 + t/4$ of the job. Additionally Tom painted $2\frac{1}{4} - t$ hours and completed $(2\frac{1}{4} - t)/3$ of the job. Thus $t/3 + t/4 + (2\frac{1}{4} - t)/3 = 1$, which yields $t = 1$.

6. The price of a diamond is proportional to the square of its mass, which is measured in carats. A six carat diamond was broken into two parts and the total price of the two pieces is $5/8$ of the price of the original diamond. What are the masses of the two pieces?

   (A) 3.5 and 2.5 (B) 5 and 1 (C) 4.5 and 1.5 (D) 4 and 2 (E) 3.6 and 2.4

   Answer: C.
   Solution. If a and $1-a$ are portions of parts, then $a^2 + (1-a)^2 = 5/8$, i.e., $a = 1/4$ or $3/4$.

7. What is the ratio of the area of a square inscribed in a semicircle to the area of a square inscribed in the entire circle?



   (A) $1/2$ (B) $2/3$ (C) $2/5$ (D) $3/4$ (E) $3/5$

   Answer: C.
   Solution. Suppose that the radius of the circle is r. Let s be the side length of the square inside the semicircle, and let t be the side length of the square inside the full circle. We want to calculate s^2/t^2. In the semicircle, there is a right triangle with sides $s, s/2$, and hypotenuse r. Thus, $r^2 = 5s^2/4$, while the full circle gives a right triangle with sides $t/2, t/2$ and hypotenuse r. So $r^2 = t^2/2$. So $s^2/t^2 = 2/5$.

8. A box contains a collection of stamps worth 23 cents and stamps worth 25 cents. The total value of the 23-cent stamps equals the total value of the 25-cent stamps, and the total value of all the stamps in the collection is less than 35 dollars. What is the maximum possible number of stamps in the box?

   (A) 96 (B) 119 (C) 121 (D) 144 (E) 192

   Answer: D.
   Solution. If x is the number of 23-cent stamps and y is the number of 25-cent stamps, then we have $23x = 25y$ and $23x + 25y < 3500$. We conclude that 23 divides y evenly and that 25 divides x evenly. Thus, if $x = 25k$, then $y = 23k$, which means that the total value of the stamps is of the form $23 \cdot 25k + 25 \cdot 23k = 1150k$ cents; moreover, we have $(25 + 23)k = 48k$ stamps. Taking $k = 3$ maximizes the total value of our stamps (at \$34.50), so our maximum number of stamps is $48 \cdot 3 = 144$.

9. In the game of dominoes, each piece is marked with two numbers. The pieces are symmetrical so that the number pair is not ordered (so, for example, $(2,6)$ is the same as $(6,2)$). How many different pieces can be formed using the numbers $1,2,\ldots,10$?

(A) 45 (B) 50 (C) 55 (D) 60 (E) 65

Answer: C.
Solution. There are $\binom{10}{2} = 10(10-1)/2 = 45$ pieces on which the two numbers do not match. There are 10 pieces on which the two numbers match. So the total number of different pieces is $45 + 10 = 55$.

10. A three-digit number is drawn at random. What is the probability that the sum of its digits is less than or equal to 3?

(A) $\frac{1}{100}$ (B) $\frac{1}{300}$ (C) $\frac{1}{90}$ (D) $\frac{2}{225}$ (E) $\frac{11}{900}$

Answer: C.
Solution. There are 900 three-digit numbers. There are 10 numbers whose digits sum to less than or equal to 3: $100, 101, 102, 110, 111, 120, 200, 201, 210, 300$. Thus, the probability is $10/900 = 1/90$

11. The lengths of the sides of a triangle are 9, 12 and 15 centimeters. What is the radius of the circumscribed circle in centimeters?

(A) 6 (B) 7 (C) 7.5 (D) 8.5 (E) 9

Answer: C.
Solution. The triangle is obtained from the right triangle whose sides are 3, 4 and 5, by a dilation of a factor of 3. By Thales's theorem, the diameter of the circumscribed circle of the original triangle is the hypotenuse. Hence the radius of the circumcircle of the enlarged triangle is $3 \times 2.5 = 7.5$.

12. Consider the parabola whose equation is $y = x^2/4$. On this curve, there are two nearest points to the point $(0,5)$. What is the sum of the y-coordinates of these two points?

(A) 5 (B) 6 (C) 7 (D) 8 (E) 9

Answer: B.
Solution. Any point on this parabola is of the form $(x, x^2/4)$. The square of the distance d between such a point and $(0,5)$ is given by

$$d^2 = (x-0)^2 + \left(\frac{x^2}{4} - 5\right)^2 = \left(\frac{x^2}{4} - 3\right)^2 + 16.$$

This is minimal when $x^2/4 = 3$, that is, $x = \pm 2\sqrt{3}$. The y coordinate of both points is 3.

13. An arithmetic sequence (a_n) satisfies $a_5 + a_6 + a_7 = 72$ and $a_{10} + a_{11} + a_{12} = 87$. Find the value of a_1.

(A) 15 (B) 16 (C) 17 (D) 18 (E) 19

Answer: E.
Solution. An entry in an arithmetic sequence is the average of the adjacent entries. Hence the first equation is equivalent to $3a_6 = 72$ and the second is equivalent to $3a_{11} = 87$. Hence $a_6 = a_1 + 5d = 24$ and $a_{11} = a_1 + 10d = 29$. The difference of these two equations gives $5d = 5$, hence $d = 1$. Now $a_6 = a_1 + 5 = 24$ implies $a_1 = 19$.

14. Which of the triangles whose side lengths are listed below has the largest area?

(A) $6, 8, 9$ (B) $6, 8, 10$ (C) $6, 8, 11$ (D) $6, 8, 12$ (E) $6, 8, 13$

Answer: B.
Solution. The area S of a triangle with sides a, b, c is $S = \frac{1}{2} ab \sin\alpha$, where α is the angle between a and b. S has a maximum for a right triangle when $a^2 + b^2 = c^2$

15. Find the value of $\dfrac{\cos 1^\circ + \cos 2^\circ + \ldots + \cos 89^\circ}{\cos 91^\circ + \cos 92^\circ + \ldots + \cos 179^\circ}$.

(A) -1 (B) 0 (C) 1 (D) $\frac{\sqrt{2}}{2}$ (E) $-\frac{\sqrt{2}}{2}$

Answer: A.
Solution. Observe that $\cos(180^\circ - \alpha) = -\cos\alpha$. That means $\cos 91^\circ = -\cos 89^\circ, \ldots, \cos 179^\circ = -\cos 1^\circ$, so that the denominator differs form the numerator by the sign minus.

16. Real estate ads suggest that 53% of homes for sale have garages, 25% have swimming pools, and 4% have both features. Let a denote the percentage that have a garage or swimming pool or both, b denotes the percentage that have a garage but not a pool, and c denotes the percentage that have a pool but not a garage. Find $b + c - a$.

(A) -4% (B) 0% (C) 4% (D) 8% (E) None is correct

Answer: A.
Solution. Observe that the collection of people who have a garage or a pool (inclusively), consists of those who have a garage but not a pool, those who have a pool but not a garage, and those who have both a garage and a pool. Furthermore, these three groups of people do not overlap. Therefore $a = b + c + 4\%$. Rearranging this, we have $b + c - a = -4\%$.

17. An integer has property Z if its digits are strictly increasing. Let N be the number of 3-digit integers from 100 to 999 that have property Z. What is the remainder of the division of N by 5?

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Answer: E.
Solution. The choice of a number with property Z is equivalent to the choice of 3 different digits out of $1, 2, \ldots, 9$, which can be done in $\binom{9}{3}$ ways. The answer is, therefore, $(9 \cdot 8 \cdot 7)/(1 \cdot 2 \cdot 3) = 84$.

18. A two-digit integer n has property X if this integer plus the integer obtained by reversing the order of its digits is a complete square. Let K be the number of two-digit integers with property X. What is the remainder of the division of K by 5?

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Answer: D.
Solution. Let the digits of n (left to right) be a and b, so that $n = 10a + b$. Then the number $s = (10a + b) + (10b + a)$ should be a complete square. We have $s = 11(a + b)$; 11 is prime, so $a + b$ should be divisible by 11. But $a + b \leqslant 18$, hence we should have $a + b = 11$. Since a and b do not exceed 9, they cannot be less than 2. Therefore, the numbers with property X are $29, 38, \ldots, 92$ - in total, 8 numbers.

19. Every evening Mr. A has a dinner with five people: Ms. B and Ms. C are young ladies, Mrs. D and Mrs. E are their mothers, and Mrs. F is Ms. B's aunt. The 6 people are seated at random at a round table. What is the probability that on a given evening at least one of Mr. A's neighbors is a young lady?

(A) 0.5 (B) 0.55 (C) 0.6 (D) 0.65 (E) 0.7

Answer: E.
Solution. Mr. A's neighbors can be chosen in $\binom{5}{2} = 10$ ways, $\binom{3}{2} = 3$ of them being unfavorable (no young ladies next to Mr. A). Hence there are $10 - 3 = 7$ favorable choices, and $P = 0.7$.

20. Three red and five blue balls are arranged in a row at random. What is the probability that the last two balls are blue?

(A) $\frac{5}{8}$ (B) $\frac{5}{14}$ (C) $\frac{1}{4}$ (D) $\frac{2}{5}$ (E) $\frac{1}{3}$

Answer: B.
Solution. Among 8! permutations of 8 distinct balls there are $\binom{5}{2} \cdot 2 \cdot 6!$ permutations with these properties. Hence the probability is $\frac{\binom{5}{2} \cdot 2 \cdot 6!}{8!} = \frac{5}{14}$.