UNC Charlotte 2013 Comprehensive Exam

March 4, 2013

1. We roll three (six-sided) dice at once. What is the probability that at least two of the dice will show the same number?

(A) $1/6$ (B) $1/3$ (C) $1/2$ (D) $4/9$ (E) $2/3$

2. Consider the equation $\log(2) + \log(\sin(\theta)) + \log(\cos(\theta)) = 0$ where θ is in radians. Which one of the following formulas describes all solutions to this equation where the "k" represents all integers?

(A) $\theta = 2k\pi + (\pi/4)$ (B) $\theta = k\pi + (\pi/4)$ (C) $\theta = k\pi + (\pi/2)$
(D) $\theta = 2k\pi \pm (\pi/4)$ (E) $\theta = k\pi \pm (\pi/4)$

3. How many ordered pairs of positive integers (x, y) satisfy the equation $\dfrac{1}{x} - \dfrac{2}{y} = \dfrac{1}{6}$?

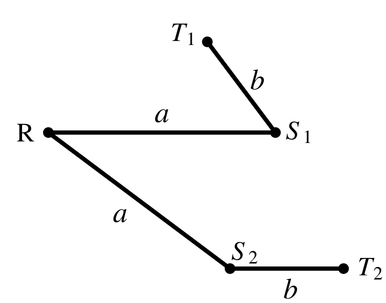
(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

4. Consider the function $f(x) = (\sin(x) - \cos(x) - 1)(\sin(x) + \cos(x) - 1)$ where $0 \le x \le 2\pi$ is measured in radians. What is the minimum value of $f(x)$?

(A) 0 (B) $-1/4$ (C) $-\sqrt{3}/4$ (D) $-1/2$ (E) $-\sqrt{2}/2$

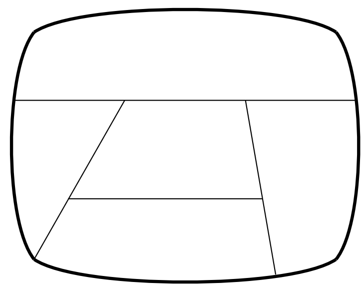
5. A robot arm in a plane has two sections, $\overline{RS}$ and $\overline{ST}$. The arm is fixed at a pivot at point R and can turn all the way around there. The arm is hinged at S and the two pieces can make any angle there. If section $\overline{RS}$ has length a and section $\overline{ST}$ has length b with $b < a$, what is the area of the region in the plane that is composed of all the points that end T of the second section of the arm can touch? [Two possible positions are shown in the figure below.]

(A) $(a^2 - b^2)\pi$
(B) $(a^2 + b^2)\pi$
(C) $ab\pi$
(D) $2ab\pi$
(E) $4ab\pi$



6. Consider the domain in the figure below, which has been separated into five regions. Four different colors (red, yellow, blue and green) are available to color the regions. The only restrictions are that each region must be entirely one color and no adjacent regions are allowed to be the same color, so at least three of the four colors must be used. How many different coloring schemes are possible?

(A) 60
(B) 72
(C) 96
(D) 100
(E) 120



7. On January 1st, there is one bean in a bin. On the 2nd, we add two beans, and then on the 3rd, we add six beans. Continuing, we add twelve beans on the 4th and thirty six beans on the 5th. Following this pattern, we continue to add beans that alternate between doubling and tripling the number added on the previous day. What is the total number of beans in the bin after we make the addition on the 24th?

(A) $(6^{12}-1)/5$ (B) $2(6^{12}-1)/5$ (C) $3(6^{12}-1)/5$ (D) $2(6^{13}-1)/5$ (E) $3(6^{13}-1)/5$

8. Mr Green sells apples for \$1.50 each at the local Farmers Market and Ms Blue sells slightly smaller apples for \$1 each. One day Ms Blue had to leave early so she asked Mr Green to manage her stall as the two were side-by-side. To make calculations easier, Mr Green mixed the apples together and changed the signs to read "5 apples for \$6". At that point they had the same number of apples left. By the end of the day he had sold all the apples, but oddly (to him) when he compared how much each would have made by selling separately and how much he had in the till, he found he was 80 dollars short. He had no clue what the problem was, so he split the money evenly and apologized to Ms Blue for messing things up. Certainly, at least one of them lost money. Did both lose money on the deal, or did one come out ahead, and how much did each lose/gain?

(A) Blue lost \$120, Green made \$40 extra (B) Blue lost \$32 and Green lost \$48
(C) Both lost \$40 (D) Blue made \$80 extra and Green lost \$160
(E) Blue made \$160 extra, Green lost \$240

9. 99 fair coins are tossed simultaneously. Let P be the probability that the number of heads is odd. Which of the following statements is true?

(A) $0 \le P < 0.2$ (B) $0.2 \le P < 0.4$ (C) $0.4 \le P < 0.6$
(D) $0.6 \le P < 0.8$ (E) $0.8 \le P \le 1$

10. In an interstellar store, a customer is buying construction materials for his granddaughter who is going to build a 6-dimensional cube. The customer only needs the edges for the cube. A big sign says edges are on sale for 2 ISD (interstellar dollar) each. The grandfather (using a cheat sheet provided by the granddaughter) asks for the correct number of edges, but is surprised at the total price. The clerk explains that even though he is buying only edges, he is required by law to also pay for the vertices necessary to build the cube. If each vertex costs 1 ISD, what did the grandfather pay?

(A) 288 ISD (B) 448 ISD (C) 576 ISD (D) 768 ISD (E) 832 ISD

11. An army is moving along in a convoy that stretches for three miles. Observing radio silence, the general at the rear of the convoy sends a message to the front via a special courier. After delivering the message, the courier returns to the rear. Both the convoy and the courier travel at (different but) constant rates. If the front (and rear) of the convoy travel six miles in the time it takes for the courier to go to the front and return to the back, what is the total distance in miles the courier travels?

(A) $5\sqrt{3}$ (B) $12 - 3\sqrt{3/4}$ (C) $3 + 3\sqrt{5}$ (D) $6\sqrt{3}$ (E) $6 + 3\sqrt{3}$

12. Consider the 2700 digit number $N = 100101102\ldots999$ obtained by listing all the three digit numbers in order. What is the remainder when N is divided by 11?

(A) 2 (B) 4 (C) 6 (D) 8 (E) 10

13. A circle is inscribed in a right triangle whose hypotenuse is 1. What is the largest possible radius of such a circle?

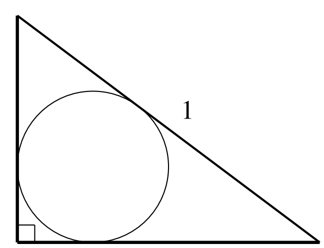
(A) $\left(\sqrt{2} - 1\right)/2$

(B) $\sqrt{2} - 1$

(C) $1/2$

(D) $\left(\sqrt{2} + 1\right)/4$

(E) $\sqrt{2}/2$



14. What is the units digit of 3^{2013}?

(A) 1 (B) 3 (C) 5 (D) 7 (E) 9

15. What is the sum of all two-digit numbers whose tens digit and units digit differ by exactly one?

(A) 878 (B) 890 (C) 900 (D) 990 (E) 991

16. Consider the following equation where m and n are real numbers:

$$(x^2 - 2x + m)(x^2 - 2x + n) = 0.$$

Suppose the four roots of the equation form an arithmetic sequence with the first (and smallest) term being $1/4$. What is the value of $|m - n|$?

(A) $3/8$ (B) $1/2$ (C) $5/8$ (D) $3/4$ (E) 1

17. On a recent trip from Northburg to Southtown, Jill decided to make a detour so she could pass through Center City. Forty minutes after she left Northburg, she noted that the remaining distance to Center City was twice as much as what she had traveled so far. After traveling another twenty one miles, she calculated that the remaining distance to Southtown was twice as much as what she had left to get to Center City. She arrived in Southtown an hour and a half later. Assuming she traveled at a constant speed, how long was this trip from Northburg to Southtown?

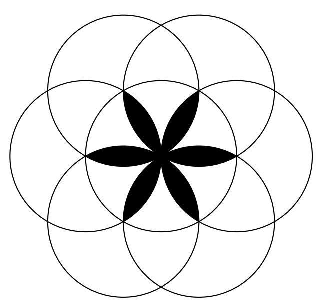
(A) 99 miles (B) 108 miles (C) 112 miles (D) 127 miles (E) 142 miles

18. How many pairs of positive integers x and y satisfy the equation $xy + 8x + y = 83$?

(A) 1 (B) 2 (C) 3 (D) 4 (E) more than 4

19. Seven circles of radius 10 are arranged as in the figure. Note that the six outer circles all pass through the center of the inner circle, the inner circle passes through the center of each outer circle, and each outer circle passes through the center of the two outer circles it is adjacent to. The area of the shaded region is A. Which of the following is true about A?

(A) $40 \le A < 60$
(B) $60 \le A < 80$
(C) $80 \le A < 100$
(D) $100 \le A < 120$
(E) $120 \le A < 140$



20. Consider the equation $\sqrt{x+3-4\sqrt{x-1}}+\sqrt{x+8-6\sqrt{x-1}}=1$ where x represents a real number. How many solutions are there?

(A) Exactly one (B) Exactly two (C) Exactly three (D) Exactly four (E) Infinitely many

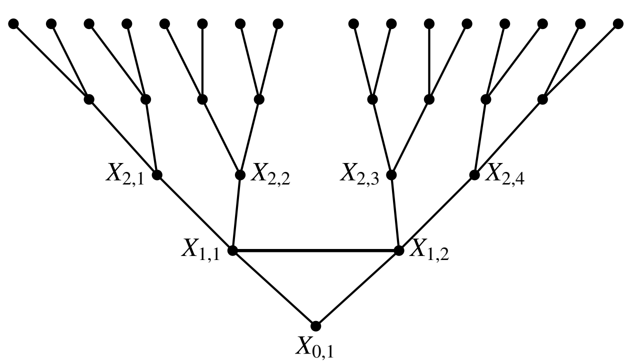
21. Every Monday, Harvey includes a puzzle on his blog. The puzzle for today goes like this: "My two sisters, all of my children and my younger brother and I were born between Jan. 1, 1901 and Dec. 31, 1999, each of us in a different year. Oddly, we all satisfy a very peculiar property. Each of us turned yx in some year $19xy$ where $0 \le y < x \le 9$ (a different year for each of us of course). And odder still, if someone born in $19ab$ satisfies this peculiar property, then one of us was born in that year. My sisters, my brother and I take care of the first four such years $19ab$, so in how many of the years $19xy$ (with $0 \le y < x \le 9$) has one of my children turned yx?"

(A) 5 (B) 6 (C) 10 (D) 12 (E) 15

22. Let $T = \{0, 1, 2, 3, 5, 7, 11\}$. How many different numbers can be obtained as the sum of three different members of T?

(A) 19 (B) 20 (C) 21 (D) 22 (E) 23

23. Consider an infinite binary tree that begins at level 0 and has an additional horizontal edge connecting the two nodes at level 1. At level n, the nodes are labeled from left to right as $X_{n,1}, X_{n,2}, \ldots, X_{n,2^n}$. How many different paths of exactly ten moves are there that start at $X_{0,1}$ and never go from an upper vertex to a lower one (but can pass back and forth on the connection between $X_{1,1}$ and $X_{1,2}$)? [The first five levels (levels 0 through 4) are shown in the figure below. The path $X_{0,1} \to X_{1,1} \to X_{1,2} \to X_{1,1} \to X_{1,2} \to X_{2,4}$ is an example of a path with five moves.]



(A) 1024
(B) 1600
(C) 2046
(D) 2048
(E) 4096

24. The Mainter brothers, Abe, Ben and Cal paint houses. They have been in the business so long that each knows exactly how many square feet he (and each of his brothers) paints in one hour and these rates never change. For their latest job, they calculated that if Abe and Ben did the job together, it would take exactly 11 hours. On the other hand, if Abe and Cal did the job together, it would take exactly 9 hours. Finally, Ben and Cal could do it in exactly 9.9 hours (or if you prefer, 9 hours and 54 minutes). They decided all three would paint this particular house. Ben and Cal started the job at 8 AM and Abe joined them at 9:00. Cal left at 1:30 and so Ben and Abe finished the job. After deducting the supply costs (paint etc.), the brothers split the net profit based on what percentage of the total square footage each painted. Who earned the most and who earned the least for this job?

(A) Abe the most, Ben the least (B) Abe the most, Cal the least
(C) Ben the most, Abe the least (D) Cal the most, Ben the least
(E) Cal the most, Abe the least

25. Surveyors are laying out a rather unusual road through a park. The park is completely flat and forms a disc of radius 20 miles. From the center C, the road is to go exactly two miles north to a point B_0, then make a 90^o left turn and go another two miles to a point B_1. At B_1 the road turns left again (not nearly as sharply), this time perpendicular to $\overline{CB_1}$. As before, it goes exactly two miles in this direction to a point B_2. This pattern is followed for the entire road – at B_k, the road makes a left turn that is perpendicular to $\overline{CB_k}$ and goes exactly two miles in this direction to B_{k+1}. Eventually the road reaches a point A (one of the B_js) that is exactly 10 miles from C. Starting from A, how many **more** two-mile segments will be needed before the road gets out of the park? [In the figure, two consecutive segments are shown starting from an arbitrary point X to the point Y and then from Y to Z.]

(A) fewer than 30
(B) between 30 and 49
(C) between 50 and 69
(D) between 70 and 89
(E) at least 90