UNC Charlotte 2013 Comprehensive Exam
SOLUTIONS

March 4, 2013

1. We roll three (six-sided) dice at once. What is the probability that at least two of the dice will show the same number?

(A) $1/6$ (B) $1/3$ (C) $1/2$ (D) $4/9$ (E) $2/3$

Solution (D): There are $6^3 = 216$ possible outcomes, each occurring with probability $1/216$. For six of these, all three numbers will be same. The number of times exactly two are the same is $3 \cdot 6 \cdot 5 = 90$: choose which die is different and that value ($3 \cdot 6 = 18$ choices for this), then choose the number that is the same for the other two die (5 choices). So the probability at least two are the same is $96/216$. Or count how many where all three are different, $6 \cdot 5 \cdot 4 = 120$, subtract this number from 216 and then divide by 216.

2. Consider the equation $\log(2) + \log(\sin(\theta)) + \log(\cos(\theta)) = 0$ where θ is in radians. Which one of the following formulas describes all solutions to this equation where the "k" represents all integers?

(A) $\theta = 2k\pi + (\pi/4)$ (B) $\theta = k\pi + (\pi/4)$ (C) $\theta = k\pi + (\pi/2)$
(D) $\theta = 2k\pi \pm (\pi/4)$ (E) $\theta = k\pi \pm (\pi/4)$

Solution (A): We have $\log(\sin(2\theta)) = \log(2\sin(\theta)\cos(\theta)) = 0$. So we need $2\theta = 2n\pi + (\pi/2)$. However, we also need θ to be in the first quadrant as both $\sin(\theta)$ and $\cos(\theta)$ must be positive. So $\theta = 2k\pi + (\pi/4)$.

3. How many ordered pairs of positive integers (x, y) satisfy the equation $\frac{1}{x} - \frac{2}{y} = \frac{1}{6}$?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Solution (D): Manipulate the equation to have equivalent forms (for nonzero integers) $6y - 12x = xy$ and $y(6 - x) = 12x$. Since both x and y are positive integers, $y > 2x \geq 2$ and $6 > x \geq 1$. Also xy is an integer multiple of 6 and $12x/(6 - x)$ is a positive integer. The latter restriction implies $x \neq 1$. But integer solutions for y exist when $x = 2, 3, 4, 5$: $x = 2$, $y = 6$; $x = 3$, $y = 12$; $x = 4$, $y = 24$; and $x = 5$, $y = 60$.

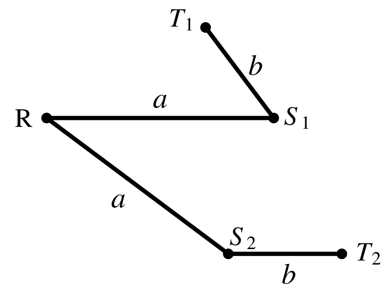
4. Consider the function $f(x) = (\sin(x) - \cos(x) - 1)(\sin(x) + \cos(x) - 1)$ where $0 \leq x \leq 2\pi$ is measured in radians. What is the minimum value of $f(x)$?

(A) 0 (B) $-1/4$ (C) $-\sqrt{3}/4$ (D) $-1/2$ (E) $-\sqrt{2}/2$

Solution (D): We have $f(x) = ((\sin(x) - 1) - \cos(x))((\sin(x) - 1) + \cos(x)) = (\sin(x) - 1)^2 - \cos^2(x) = \sin^2(x) - 2\sin(x) + 1 - \cos^2(x) = 2\sin^2(x) - 2\sin(x)$. Setting $z = \sin(x)$, we have a parabola $y = 2(z^2 - z)$ that opens up. The vertex is at $(1/2, -1/2)$.

5. A robot arm in a plane has two sections, $\overline{RS}$ and $\overline{ST}$. The arm is fixed at a pivot at point R and can turn all the way around there. The arm is hinged at S and the two pieces can make any angle there. If section $\overline{RS}$ has length a and section $\overline{ST}$ has length b with $b < a$, what is the area of the region in the plane that is composed of all the points that end T of the second section of the arm can touch? [Two possible positions are shown in the figure below.]

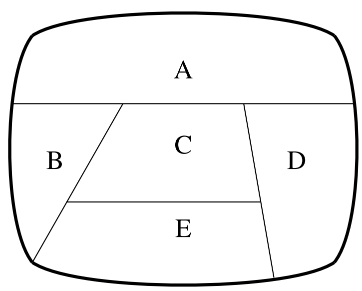
(A) $(a^2 - b^2)\pi$
(B) $(a^2 + b^2)\pi$
(C) $ab\pi$
(D) $2ab\pi$
(E) $4ab\pi$



Solution (E): The region that T touchs is the annulus formed by the circle with center at R of radius $a + b$ and the one with center R and radius $a - b$. So the total area is $\pi(a + b)^2 - \pi(a - b)^2 = 4ab\pi$.

6. Consider the domain in the figure below, which has been separated into five regions. Four different colors (red, yellow, blue and green) are available to color the regions. The only restrictions are that each region must be entirely one color and no adjacent regions are allowed to be the same color, so at least three of the four colors must be used. How many different coloring schemes are possible?

(A) 60
(B) 72
(C) 96
(D) 100
(E) 120



Solution (B): To start, choose colors for regions A, B and C, all three will have to be different. Since regions A and E do not touch, they can be the same color. In this case, region D can either be the same color as region B or it could be colored with the fourth color. So there are $4 \cdot 3 \cdot 2 \cdot 2$ ways do the coloring this way. If E and A are different colors, then all four colors have been used and now region D must be the same color as region B. For this scheme there are $4 \cdot 3 \cdot 2$ ways to color. Thus the total is $48 + 24 = 72$.

7. On January 1st, there is one bean in a bin. On the 2nd, we add two beans, and then on the 3rd, we add six beans. Continuing, we add twelve beans on the 4th and thirty six beans on the 5th. Following this pattern, we continue to add beans that alternate between doubling and tripling the number added on the previous day. What is the total number of beans in the bin after we make the addition on the 24th?

(A) $(6^{12}-1)/5$ (B) $2(6^{12}-1)/5$ (C) $3(6^{12}-1)/5$ (D) $2(6^{13}-1)/5$ (E) $3(6^{13}-1)/5$

Solution (C): We have $1, 1+2, 1+2+6, 1+2+6+12, 1+2+6+12+36, 1+2+6+12+36+72$... Condense the sequence by looking at only the even days, combining the additions as follows: $3, 3+3\cdot 6, 3+3\cdot 6+3\cdot 36$, $3 + 3 \cdot 6 + 3 \cdot 36 + 3 \cdot 216$. On the 24th, we have $3 + 3 \cdot 6 + 3 \cdot 36 + 3 \cdot 216 + \cdots + 3 \cdot 6^{11} = 3(6^{12} - 1)/(6 - 1)$.

8. Mr Green sells apples for \$1.50 each at the local Farmers Market and Ms Blue sells slightly smaller apples for \$1 each. One day Ms Blue had to leave early so she asked Mr Green to manage her stall as the two were side-by-side. To make calculations easier, Mr Green mixed the apples together and changed the signs to read “5 apples for \$6”. At that point they had the same number of apples left. By the end of the day he had sold all the apples, but oddly (to him) when he compared how much each would have made by selling separately and how much he had in the till, he found he was 80 dollars short. He had no clue what the problem was, so he split the money evenly and apologized to Ms Blue for messing things up. Certainly, at least one of them lost money. Did both lose money on the deal, or did one come out ahead, and how much did each lose/gain?

(A) Blue lost \$120, Green made \$40 extra (B) Blue lost \$32 and Green lost \$48
(C) Both lost \$40 (D) Blue made \$80 extra and Green lost \$160
(E) Blue made \$160 extra, Green lost \$240

Solution (E): Let x denote the number of apples each has for sale. Then Ms Blue should receive x dollars and Mr Green $1.5x$ dollars. By mixing and selling $2x$ apples at 5 for \$6, Mr Green is effectively charging \$1.20 per apple. At this rate he is losing \$0.30 an apple and Ms Blue is receiving an extra \$0.20 per apple. As he is \$80 short, $80 = 0.3x - 0.2x = 0.1x$ and we have $x = 800$. At the end of the day, Ms Blue made an extra \$160 and Mr Green lost \$240.

9. 99 fair coins are tossed simultaneously. Let P be the probability that the number of heads is odd. Which of the following statements is true?

(A) $0 \le P < 0.2$ (B) $0.2 \le P < 0.4$ (C) $0.4 \le P < 0.6$
(D) $0.6 \le P < 0.8$ (E) $0.8 \le P \le 1$

Solution (C): Since there is an odd number of coins, either the number of heads is odd and the number of tails is even, or the number of heads is even and the number of tails is odd. Since the coin is fair, half of the time the number of heads will be odd.

10. In an interstellar store, a customer is buying construction materials for his granddaughter who is going to build a 6-dimensional cube. The customer only needs the edges for the cube. A big sign says edges are on sale for 2 ISD (interstellar dollar) each. The grandfather (using a cheat sheet provided by the granddaughter) asks for the correct number of edges, but is surprised at the total price. The clerk explains that even though he is buying only edges, he is required by law to also pay for the vertices necessary to build the cube. If each vertex costs 1 ISD, what did the grandfather pay?

(A) 288 ISD (B) 448 ISD (C) 576 ISD (D) 768 ISD (E) 832 ISD

Solution (B): Just as the points $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(1, 1, 0)$, $(1, 0, 1)$, $(0, 1, 1)$ and $(1, 1, 1)$ are the vertices of a three-dimensional cube, the points of the form $(a_1, a_2, a_3, a_4, a_5, a_6)$ where each $a_i \in \{0, 1\}$ are the vertices of a six-dimensional cube (in six-dimensional space). There are $2^6 = 64$ vertices. An edge connects a pair of vertices if and only if the vertices have exactly one coordinate that is different. So each vertex is connected by an edge to six other vertices. Thus the total number of edges is $6 \cdot 2^6/2 = 192$. The total price is $2 \cdot 192 + 64$.

11. An army is moving along in a convoy that stretches for three miles. Observing radio silence, the general at the rear of the convoy sends a message to the front via a special courier. After delivering the message, the courier returns to the rear. Both the convoy and the courier travel at (different but) constant rates. If the front (and rear) of the convoy travel six miles in the time it takes for the courier to go to the front and return to the back, what is the total distance in miles the courier travels?

(A) $5\sqrt{3}$ (B) $12 - 3\sqrt{3/4}$ (C) $3 + 3\sqrt{5}$ (D) $6\sqrt{3}$ (E) $6 + 3\sqrt{3}$

Solution (C): Let q be the distance traveled by the courier to get to the front of the convoy and let p be the distance traveled by the convoy during the same time. We have a relation: $q = p + 3$. To reach the rear of convoy when the rear has advanced 6 miles from the starting point, the courier needs to travel $q - 6$ miles back (after reaching the front). As a fraction of the forward journey, we have $(q-6)/q$. Thus the convoy will move $p(q-6)/q$ while the courier is returning to the rear. We have $6 = p+p(q-6)/q = q-3+(q-3)(q-6)/q$.

Multiply both sides by q, and rearrange to get $q^2 - 9q + 9 = 0$. Thus $q = \left(9 + 3\sqrt{5}\right)/2$. The total distance the courier travels: $2q - 6 = 3 + 3\sqrt{5}$.

12. Consider the 2700 digit number $N = 100101102\dots999$ obtained by listing all the three digit numbers in order. What is the remainder when N is divided by 11?

(A) 2 (B) 4 (C) 6 (D) 8 (E) 10

Solution (E): The number can be written as $998999 + 996997\cdot 10^6 + 994995\cdot 10^{12} + \cdots 100101\cdot 10^{6\cdot 449}$. All of the powers 10^{6k} are 1 more than a multiple of 11 as are all of the six digit numbers. So N is $450 = 440 + 10$ larger than a multiple of 11. Thus the remainder is 10.

13. A circle is inscribed in a right triangle whose hypotenuse is 1. What is the largest possible radius of such a circle?

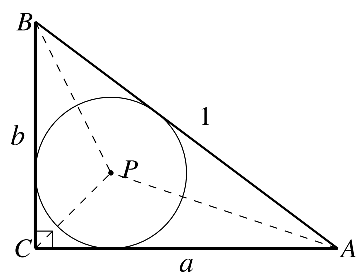
(A) $\left(\sqrt{2} - 1\right)/2$

(B) $\sqrt{2} - 1$

(C) $1/2$

(D) $\left(\sqrt{2} + 1\right)/4$

(E) $\sqrt{2}/2$



Solution (A): Let r be the radius and P be the center of the inscribed circle. Then the triangle can be subdivided into three triangles using P and the three pairs $\{A, B\}$, $\{A, C\}$ and $\{B, C\}$. The sum of the areas of these three triangles is $r/2 + ra/2 + rb/2 = ab/2$. Solving for r we have $r = ab/(a + b + 1)$. If you suspect the largest value for the fraction on the right is when $a = b = \sqrt{2}/2$, you are correct. The form in (A) is obtained by rationalizing the denominator. To prove that $a = b$ gives the maximum radius takes a bit more work.

Suppose $F(x) = g(x)/h(x)$ is a variable fraction where both numerator and denominator are positive. If $h(x)$ takes its smallest value when $x = t$ and $g(x)$ takes its largest value when $x = t$, then $F(t)$ is the largest value of $F(x)$. Unfortunately, in the present form, the numerator ab is as large as possible at the same time the denominator $a + b + 1$ is also as large as possible (noting that $b = \sqrt{1 - a^2}$). To fix that, multiply top and bottom by $1/ab$ to get $r = 1/((1/b) + (1/a) + (1/ab))$. Now all we have to do is show the smallest value for the denominator is when $a = b$. We have $a = \cos(\alpha)$ and $b = \sin(\alpha)$ where α is the angle at A. Thus $ab = \sin(2\alpha)/2$. Certainly, the largest value of ab occurs when $\alpha = 45^o$ and so when $a = b = \sqrt{2}/2$. So the smallest value of $1/ab$ is 2. As for $(1/b) + (1/a)$, it is as small as possible at the same time as its square is as small as possible. Squaring yields $(1/b^2) + (2/ab) + (1/a^2)$ and $(1/b^2) + (1/a^2) = (a^2 + b^2)/a^2b^2 = (1/ab)^2$. So $2/ab$ and the sum $(1/b^2) + (1/a^2)$ are also as small as possible when $a = b$.

14. What is the units digit of 3^{2013}?

(A) 1 (B) 3 (C) 5 (D) 7 (E) 9

Solution (B): Note that $3^4 = 81$, so the units digit of 3^{4k} is a 1 for each positive integer k. We can rewrite the product as $3 \cdot 3^{2012} = 3 \cdot (3^4)^{503}$. Thus the units digit is a 3.

15. What is the sum of all two-digit numbers whose tens digit and units digit differ by exactly one?

(A) 878 (B) 890 (C) 900 (D) 990 (E) 991

Solution (B): The numbers are 10, 12, 21, 23, 32, 34, ..., 78, 87, 89, 98. So the sum of all of them can be split into the sum $10 + 21 + 32 + \cdots + 98 = 9(108)/2$ plus the sum $12 + 23 + \cdots + 89 = 8(101)/2$.

16. Consider the following equation where m and n are real numbers:

$$(x^2 - 2x + m)(x^2 - 2x + n) = 0.$$

Suppose the four roots of the equation form an arithmetic sequence with the first (and smallest) term being $1/4$. What is the value of $|m - n|$?

(A) 3/8 (B) 1/2 (C) 5/8 (D) 3/4 (E) 1

Solution (B): The four roots are 0.25, $k + 0.25$, $2k + 0.25$, $3k + 0.25$. Since the coefficient on x is 2 in both factors, one of the two quadratic expressions factors as $(x - 0.25)(x - (3k + 0.25))$ and the other as $(x - (k + 0.25))(x - (2k + 0.25))$. In addition, $3k + 0.5 = 2$ so $k = 1/2$. If we let $m = (k + 0.25)(2k + 0.25)$ and $n = 0.25(3k + 0.25)$, we have $m - n = 15/16 - 7/16 = 1/2$.

17. On a recent trip from Northburg to Southtown, Jill decided to make a detour so she could pass through Center City. Forty minutes after she left Northburg, she noted that the remaining distance to Center City was twice as much as what she had traveled so far. After traveling another twenty one miles, she calculated that the remaining distance to Southtown was twice as much as what she had left to get to Center City. She arrived in Southtown an hour and a half later. Assuming she traveled at a constant speed, how long was this trip from Northburg to Southtown?

(A) 99 miles (B) 108 miles (C) 112 miles (D) 127 miles (E) 142 miles

Solution (A): Let x denote the number of miles Jill had traveled in the first 40 minutes. Since this was one third of the way to Center City, it took her a total of 2 hours to get to Center City. Next let y be the number of miles she had left to get to Center City after she had gone the additional 21 miles. We have $2x = 21 + y$, and the total trip is $3x + y = x + 2y + 21$. She can travel $1.5x = 4y/3$ miles in 1 hour. Thus $9x = 8y$. Solve to get $x = 24$ and $y = 27$. The total distance is 99 miles.

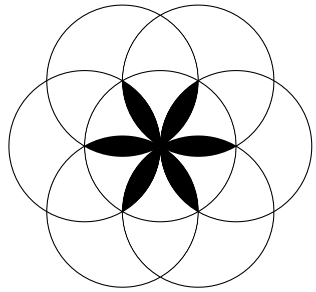
18. How many pairs of positive integers x and y satisfy the equation $xy + 8x + y = 83$?

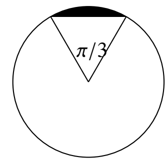
(A) 1 (B) 2 (C) 3 (D) 4 (E) more than 4

Solution (A): We can rewrite the equation as $x(y + 8) = 83 - y$. The factor $y + 8$ is at least 9, so $1 \le x \le 9$. An alternate form for the equation is $y(x + 1) = 83 - 8x$. The right hand side is odd, so both $x + 1$ and y are odd integers. Now simply try $x = 2$, $x = 4$, $x = 6$, $x = 8$. For $x = 2$: $3y = 83 - 16 = 67$, no integer solution. For $x = 4$: $5y = 83 - 40 = 43$, again no integer solution. For $x = 6$, $7y = 83 - 48 = 35$, so $y = 5$. Finally for $x = 8$, $9y = 83 - 64 = 19$, no solution. An alternate way to solve is to add 8 to both sides of the original equation to get $xy + 8x + y + 8 = 91$. Factor to get $(x + 1)(y + 8) = 7 \cdot 13$. It must be that $x + 1 = 7$ and $y + 8 = 13$.

19. Seven circles of radius 10 are arranged as in the figure. Note that the six outer circles all pass through the center of the inner circle, the inner circle passes through the center of each outer circle, and each outer circle passes through the center of the two outer circles it is adjacent to. The area of the shaded region is A. Which of the following is true about A?

(A) $40 \le A < 60$
(B) $60 \le A < 80$
(C) $80 \le A < 100$
(D) $100 \le A < 120$
(E) $120 \le A < 140$





Solution (D): We may assume the inner circle has center at $(0, 0)$ and for the outer circles, the centers are at angles 0, π, $\pm\pi/3$ and $\pm 2\pi/3$, 10 units from $(0, 0)$. The colored portion is made up of twelve identical pieces: start with the wedge of a circle where the angle is $\pi/3$, connect the secant line and then remove the equilateral triangle (in this case with side length 10) formed. The remaining area is $100\left(\pi/6 - \sqrt{3}/4\right)$. So the total area is $12 \cdot 100\left(\pi/6 - \sqrt{3}/4\right) = 100\left(2\pi - 3\sqrt{3}\right)$. To estimate the area, note that $17^2 = 289$, and $17.5^2 = 306.25$ ($18^2 = 324$), so $1.7 < \sqrt{3} < 1.75$. Thus the area is between 100 and 120.

20. Consider the equation $\sqrt{x + 3 - 4\sqrt{x-1}} + \sqrt{x + 8 - 6\sqrt{x-1}} = 1$ where x represents a real number. How many solutions are there?

(A) Exactly one (B) Exactly two (C) Exactly three
(D) Exactly four (E) Infinitely many

Solution (E): Make the substitution $z = \sqrt{x-1}$. Then the equation becomes $\sqrt{z^2 + 4 - 4z} + \sqrt{z^2 + 9 - 6z} = 1$. Next note that $z^2 - 4z + 4 = (z-2)^2$ and $z^2 - 6z + 9 = (z-3)^2$. Thus the equation reduces to $|z-2| + |z-3| = 1$. If $2 \le z \le 3$, then $|z-2| + |z-3| = z - 2 + 3 - z = 1$. So all x in the interval $5 \le x \le 10$ are solutions.

21. Every Monday, Harvey includes a puzzle on his blog. The puzzle for today goes like this: "My two sisters, all of my children and my younger brother and I were born between Jan. 1, 1901 and Dec. 31, 1999, each of us in a different year. Oddly, we all satisfy a very peculiar property. Each of us turned yx in some year $19xy$ where $0 \le y < x \le 9$ (a different year for each of us of course). And odder still, if someone born in $19ab$ satisfies this peculiar property, then one of us was born in that year. My sisters, my brother and I take care of the first four such years $19ab$, so in how many of the years $19xy$ (with $0 \le y < x \le 9$) has one of my children turned yx?"

(A) 5 (B) 6 (C) 10 (D) 12 (E) 15

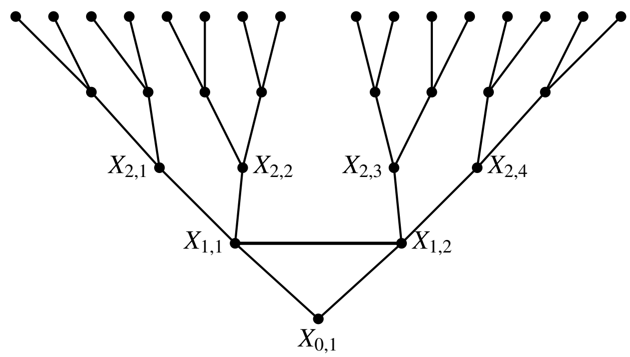
Solution (E): Consider all years $19xy$ with $9 \ge x > y \ge 0$ and subtract yx to see what year $19ab$ would be the birth year. Starting with $x = 1$, $y = 0$, a person who turned 01 in 1910 was born in 1909. That person will turn 12 in 1921, 23 in 1932, etc. To be 02 in 1920 means a birth year of 1918. The person who turned 03 in 1930 was born in 1927. Next is the one who turned 04 in 1940 and was born 1936. So the oldest child was born in 1945 and turned 05 in 1950. This child turned 16 in 1961, 27 in 1972, 38 in 1983 and 49 in 1994 for a total of five years where the child's age was yx in $19xy$. The next oldest turned 06 in 1960, so was born in 1954. The special birthdays are at 17 in 1971, 28 in 1982 and 39 in 1993. The middle child turned 07 in 1970, then was 18 in 1982 and 29 in 1992. Child number four turned 08 in 1980 and was 19 in 1991. Finally the youngest child turned 09 in 1990 and was born in 1981. There are $5 + 4 + 3 + 2 + 1 = 15$ years where a child turned yx in $19xy$.

22. Let $T = \{0, 1, 2, 3, 5, 7, 11\}$. How many different numbers can be obtained as the sum of three different members of T?

(A) 19 (B) 20 (C) 21 (D) 22 (E) 23

Solution (B): The smallest possible sum is $3 = 0 + 1 + 2$ and the largest is $23 = 5 + 7 + 11$. There are twenty one integers in this range, but it is impossible to have a sum of 22. Except for 22, all other sums are possible for a total of twenty.

23. Consider an infinite binary tree that begins at level 0 and has an additional horizontal edge connecting the two nodes at level 1. At level n, the nodes are labeled from left to right as $X_{n,1}, X_{n,2}, \ldots, X_{n,2^n}$. How many different paths of exactly ten moves are there that start at $X_{0,1}$ and never go from an upper vertex to a lower one (but can pass back and forth on the connection between $X_{1,1}$ and $X_{1,2}$)? [The first five levels (levels 0 through 4) are shown in the figure below. The path $X_{0,1} \to X_{1,1} \to X_{1,2} \to X_{1,1} \to X_{1,2} \to X_{2,4}$ is an example of a path with five moves.]



(A) 1024

(B) 1600

(C) 2046

(D) 2048

(E) 4096

Solution (C): For each node $X_{i,j}$ for $1 \le i \le 10$, there is a unique path with 10 moves from $X_{0,1}$ to $X_{i,j}$. For example the only path with 10 moves from $X_{0,1}$ to $X_{6,1}$ is

$$X_{0,1} \to X_{1,1} \to X_{1,2} \to X_{1,1} \to X_{1,2} \to X_{1,1} \to X_{2,1} \to X_{3,1} \to X_{4,1} \to X_{5,1} \to X_{6,1}.$$

The number of nodes on levels 1 through 10 is $2 + 4 + \cdots + 2^{10} = 2(2^{10} - 1) = 2046$.

24. The Mainter brothers, Abe, Ben and Cal paint houses. They have been in the business so long that each knows exactly how many square feet he (and each of his brothers) paints in one hour and these rates never change. For their latest job, they calculated that if Abe and Ben did the job together, it would take exactly 11 hours. On the other hand, if Abe and Cal did the job together, it would take exactly 9 hours. Finally, Ben and Cal could do it in exactly 9.9 hours (or if you prefer, 9 hours and 54 minutes). They decided all three would paint this particular house. Ben and Cal started the job at 8 AM and Abe joined them at 9:00. Cal left at 1:30 and so Ben and Abe finished the job. After deducting the supply costs (paint etc.), the brothers split the net profit based on what percentage of the total square footage each painted. Who earned the most and who earned the least for this job?

(A) Abe the most, Ben the least (B) Abe the most, Cal the least

(C) Ben the most, Abe the least (D) Cal the most, Ben the least

(E) Cal the most, Abe the least

Solution (A): Let a be the rate at which Abe paints, b the rate Ben paints and c the rate Cal paints. Also let H be the total number of square feet of painting to do in the house. Then $11a + 11b = H$, $9a + 9c = H$ and $9.9b + 9.9c = H$. An equivalent system is $a + b = H/11$, $a + c = H/9$ and $b + c = H/9.9$. Solving this systems yields $a = H(1/9 + 1/11 - 1/9.9) = 5H/99$, $b = H(1/11 + 1/9.9 - 1/9) = 4H/99$ and $c = H(1/9 + 1/9.9 - 1/11) = 6H/99$. Cal paints for 5.5 hours, so he paints exactly 1/3 of the house. Abe paints in 4 hours what Ben paints in 5. So by 1:30, Abe has painted more of the house than Ben. Since Abe paints faster than Ben and Cal painted exactly one third of the house, Abe painted the most and Ben the least.

25. Surveyors are laying out a rather unusual road through a park. The park is completely flat and forms a disc of radius 20 miles. From the center C, the road is to go exactly two miles north to a point B_0, then make a 90^o left turn and go another two miles to a point B_1. At B_1 the road turns left again (not nearly as sharply), this time perpendicular to $\overline{CB_1}$. As before, it goes exactly two miles in this direction to a point B_2. This pattern is followed for the entire road – at B_k, the road makes a left turn that is perpendicular to $\overline{CB_k}$ and goes exactly two miles in this direction to B_{k+1}. Eventually the road reaches a point A (one of the B_js) that is exactly 10 miles from C. Starting from A, how many **more** two-mile segments will be needed before the road gets out of the park? [In the figure, two consecutive segments are shown starting from an arbitrary point X to the point Y and then from Y to Z.]

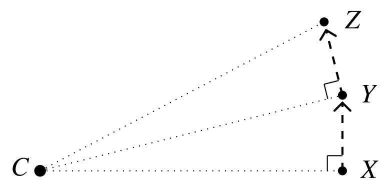
(A) fewer than 30

(B) between 30 and 49

(C) between 50 and 69

(D) between 70 and 89

(E) at least 90



Solution (D): If the square of the length of $\overline{CB_k}$ is q, then the square of the length of $\overline{CB_{k+1}}$ is $q + 4$. To have B_{k+1} outside the circle of radius 20 with center C, we need $q + 4$ to be greater than 400. The sequence of distances from $A = B_j$ on runs as follows: $\overline{CB_j} = \sqrt{100}$, $\overline{CB_{j+1}} = \sqrt{104}$, $\overline{CB_{j+2}} = \sqrt{108}$. In general, $\overline{CB_{j+i}} = \sqrt{100 + 4i}$. To have B_{j+i} outside we need $100 + 4i > 400$, so $i > 75$ (if $i = 75$, B_{j+i} is on the park boundary).