UNIVERSITY OF NORTH CAROLINA CHARLOTTE
1997 HIGH SCHOOL MATHEMATICS CONTEST
March 10, 1997
Problems with solutions

1. Which of the five fractions is largest?

(A) $\frac{25038876541}{25038876543}$ **(B)** $\frac{25038876543}{25038876545}$ **(C)** $\frac{25038876545}{25038876547}$

(D) $\frac{25038876547}{25038876549}$ **(E)** $\frac{25038876549}{25038876551}$

(E) The numbers are all of the form $\frac{n}{n+2}$ for n equal to five consecutive odd numbers. Since $\frac{n}{n+2}$ gets larger as n gets larger, the fraction E is the largest.

2. An eight-bit binary word is a sequence of eight digits each of which is either 0 or 1. The number of different eight-bit binary words is

(A) 32 **(B)** 64 **(C)** 128 **(D)** 256 **(E)** 512

(D) Since there are two choices for each bit, the number of different words is $2^8 = 256$.

3. If $P(x) = 2x^3 + kx^2 + x$, find k such that $x - 1$ is a factor of $P(x)$.

(A) -3 **(B)** $-1/3$ **(C)** 0 **(D)** 1/3 **(E)** 3

(A) The value of k that works is the one for which $P(1) = 0$ since $x - r$ is a factor of P exactly when $P(r) = 0$. So we have $P(1) = 2 + k + 1 = 0$, and $k = -3$.

4. If $\log_2 x + \log_2 5 = \log_2 x^2 - \log_2 14$, then $x =$

(A) 0 **(B)** 70 **(C)** both 0 and 70 **(D)** $\log_2 70$ **(E)** 2^{70}

(B) The given equation can be transformed to get

$$5x = \frac{x^2}{14}$$

which can be solved for x to yield $x = 70$. Since $\log_2 0$ is undefined, 0 is not a possible value of x.

5. Which of the following could be the exact value of n^4, where n is a positive integer?

(A) 1.6×10^{20} **(B)** 1.6×10^{21} **(C)** 1.6×10^{22} **(D)** 1.6×10^{23} **(E)** 1.6×10^{24}

(B) Only $1.6 \times 10^{21} = 16 \times 10^{20}$ is a perfect fourth power of an integer.

6. The following table gives the distribution of families in the town of Colville in 1991 by the number of children. If there were 5000 families, how many families had no children.

Number of Children	0	1	2	3	4 or more
Percent of Families	n%	19%	18%	10%	9%

(A) 2000 **(B)** 2200 **(C)** 2350 **(D)** 2500 **(E)** 2800

(B) Exactly $19 + 18 + 10 + 9 = 56$ percent of the families in the table have one or more children. Therefore, $n = 100 - 56 = 44$ and 44% of the families have no children. Finally, $.44 \times 5,000 = 2,200$.

7. A chemist has a solution consisting of 5 ounces of propanol and 17 ounces of water. She would like to change the solution into a 40% propanol solution by adding z ounces of propanol. Which of the following equations should she solve in order to determine the value of z?

(A) $\frac{5}{z+17} = \frac{40}{100}$ **(B)** $\frac{z+5}{22} = \frac{40}{100}$ **(C)** $\frac{z+5}{17} = \frac{40}{100}$

(D) $\frac{z+5}{z+17} = \frac{40}{100}$ **(E)** $\frac{z+5}{z+22} = \frac{40}{100}$

(E) The new volume must be $22 + z$ and the amount of propanol is $5 + z$, so the fraction $(z + 5)/(z + 22)$ must be 0.4.

8. How many integers n satisfy $|n^3 - 222| < 888$?

(A) 11 **(B)** 17 **(C)** 18 **(D)** 19 **(E)** 20

(D) The given equation is equivalent to $-888 < n^3 - 222 < 888$ which in turn is eqivalent to $-666 < n^3 < 1110$, which has the integral solutions $-8, -7, -6, \ldots, 0, 1, 2, \ldots, 10$ of which there are 19.

9. Seven women and five men attend a party. At this party each man shakes hands with each other person once. Each woman shakes hands only with men. How many handshakes took place at the party?

(A) 31 **(B)** 35 **(C)** 45 **(D)** 56 **(E)** 66

(C) There are $5 \cdot 7 = 35$ handshakes between men and women, and $\binom{5}{2} = 10$ handshakes exchanged among the men, for a total of 45 handshakes.

10. Suppose $ab < 0$. Which of the following points could **not** satisfy $y = ax + b$?

(A) $(0, 1)$ **(B)** $(1, 0)$ **(C)** $(-1, 0)$ **(D)** $(0, -1)$ **(E)** $(1, 1)$

(C) The lines $y = x - 1, y = -x + 1$, and $y = -x + 2$ contain the points $(0, 1), (1, 0), (0, -1)$, and $(1, 1)$, but $(-1, 0)$ cannot belong to such a line. If a line with positive slope contains $(-1, 0)$, that line must also have a positive y-intercept, and if a line with negative slope contains $(-1, 0)$, that line must also have a negative y-intercept.

11. Let x and y be two real numbers satisfying $x + y = 6$ and $xy = 7$. Find the value of $x^3 + y^3$.

(A) 55 **(B)** 62 **(C)** 78 **(D)** 90 **(E)** 216

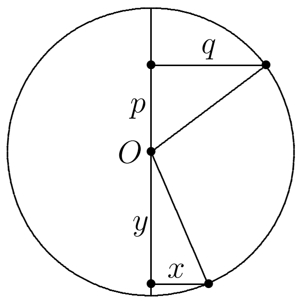
(D) Expand $(x + y)^3$ to yield

$$\begin{aligned} 216 = 6^3 &= (x+y)^3 \\ &= x^3 + 3x^2y + 3xy^2 + y^3 \\ &= x^3 + y^3 + 3xy(x+y) \\ &= x^3 + y^3 + 3 \cdot 7 \cdot 6 \\ &= x^3 + y^3 + 126. \end{aligned}$$

So $x^3 + y^3 = 216 - 126 = 90$.

12. In the figure below, the two triangles are right triangles with sides of lengths x, y, p, and q, as shown. Given that $x^2+y^2+p^2+q^2 = 72$, find the circumference of the circle.

(A) 8π **(B)** 9π **(C)** 12π **(D)** 24π **(E)** 36π



(C) Note that $x^2 + y^2 + p^2 + q^2 = 2r^2 = 72$, so $r^2 = 36$, and $r = 6$. Hence the circumference is 12π.

13. Three integers a, b and c have a product of $9!$ and satisfy $a \le b \le c$. What is the smallest possible value of $c - a$?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 42 **(E)** 51

(C) From the factorization $9! = 2^7 \cdot 3^4 \cdot 5 \cdot 7$, we see that a, b and c must have prime factors among the primes $2, 3, 5,$ and 7. Because the cube root of $9!$ is close to 70, and $a \cdot b \cdot c = 9!$, it follows that $c - a$ is smallest when a and c are close to 70. Trying $2 \cdot 5 \cdot 7$ for one of the factors leaves $2^6 \cdot 3^4 = 72^2$. Thus $c - a \le 72 - 70 = 2$. To see that no smaller difference is possible, note that $9!$ is not a perfect cube so $c - a$ cannot be zero. If $c - a = 1$, then there is an integer n such that either $n \cdot (n+1)^2 = 9!$ or $n^2 \cdot (n+1) = 9!$. But $n = 70$ makes these two left sides too small, and $n = 71$ makes them both too big.

14. A $4 \times 4 \times 4$ cube is made from 32 white unit cubes and 32 black unit cubes. What is the largest possible percent of black surface area?

(A) 50% **(B)** 60% **(C)** 64% **(D)** $66\frac{2}{3}$% **(E)** 75%

> **(E)** Put all the black unit cubes at the corners and along the edges of the big cube so they contribute either 3 or 2 square units to the surface. There are just enough black unit cubes to fill all eight corner positions and all the other $12 \times 2 = 24$ edge positions, so each face of the big cube is three-forths covered by black unit squares.

15. Suppose a, b and c are real numbers for which

$$\frac{a}{b} > 1 \text{ and } \frac{a}{c} < 0.$$

Which of the following must be correct?

(A) $a + b - c > 0$ **(B)** $a > b$ **(C)** $(a - c)(b - c) > 0$

(D) $a + b + c > 0$ **(E)** $abc > 0$

> **(C)** First, note that a and b agree in signs and that c has a different sign. Thus, a, b and $-c$ have the same sign. Therefore, $a - c$ and $b - c$ have the same sign. Hence their product must be positive. None of the other inequalities need be true.

16. The product of the digits of Ashley's age is the same nonzero number as it was six years ago. In how many years will it be the same again?

(A) 14 **(B)** 18 **(C)** 19 **(D)** 24 **(E)** 26

> **(B)** Suppose Ashley is $10u+v$ years old. Then $uv = (u-1)(v+4)$. Clearing uv, we get $u = 2$ and $v = 4$. It will be another 18 years before the product of the digits of Ashley's age is again 8.

17. It takes Mathias and Anders 1188 hours to paint the Gaffney Peach. It takes Anders and Tellis 1540 hours to paint the peach; for Tellis and Hal, it takes 1890 hours; and for Hal and Mathias, it takes 1386 hours. How long would it take all four of them working together to paint the peach?

(A) 364.7 hours **(B)** 412.3 hours **(C)** 670.7 hours

(D) 729.5 hours **(E)** 824.6 hours

(D) There is more information given than is needed. Let m, a, t, h denote the rates, in jobs per hour, of Mathias, Anders, Tellis and Hal respectively. Then $(m + a) \cdot 1188 = 1$ and $(t + h) \cdot 1890 = 1$. From this it follows that $1 = (m + a + t + h)x = (\frac{1}{1188} + \frac{1}{1890}) \cdot 1$ and from this we get $x = 729.473 \ldots \approx 729.5$ hours.

18. The number $N = 700,245$ can be expressed as the product of three two-digit integers, $x, y,$ and z. What is $x + y + z$?

(A) 210 **(B)** 267 **(C)** 269 **(D)** 271 **(E)** 272

(B) Factor N into primes to get $N = 3^4 \cdot 5 \cdot 7 \cdot 13 \cdot 19$. The only way to divide these primes into three groups each with product less than 100 is $3^4, 5 \cdot 19,$ and $7 \cdot 13$ since the 19 cannot be matched with a 3 ($3 \cdot 19 = 57$ and $57 \cdot 99 \cdot 99 < N$), so the sum is $81 + 95 + 91 = 267$.

19. In how many points can a line intersect the graph of the function $f(x) = x^2 \sin(x)$?

I. no points

II. one point

III. infinitely many points

(A) II only **(B)** III only **(C)** I and II only **(D)** I and III only

(E) II and III only

(E) The x-axis intersects the graph infinitely many times and vertical lines intersect just once. No line in the plane is disjoint from the graph of f.

20. Cifarelli Builders has just completed developing a section of homes in Southeast Charlotte. The homes are numbered consecutively starting with the address 1. The contractor in charge of ordering the single-digit brass numerals that will be used on each house for its address has determined that 999 numerals need to be ordered. How many homes are in the development?

(A) 200 **(B)** 369 **(C)** 379 **(D)** 381 **(E)** 999

(B) Construct a table.

House number	**#of digits used**
1-9	$9 \cdot 1 = 9$
10-99	$90 \cdot 2 = 180$
100-199	$100 \cdot 3 = 300$
200-299	$100 \cdot 3 = 300$
300-399	$100 \cdot 3 = 300$

The first four rows show that 789 of the 999 digits are used on the first 299 houses. That leaves 210 more digits to be used on the remaining houses. This is just enough for 70 more houses. So there are $299 + 70 = 369$ houses in the development.

21. Eight people Amy, Bee, Cindy, Dennis, Eli, Fay, Gil, and Hilary attend a dinner party. They need to be seated around a circular table, but Cindy and Gil, the hosts, choose not to be seated next to one another. How many different arrangements are there which seat Gil in the seat nearest the kitchen (so he can serve the dinner)?

(A) 3600 **(B)** 4320 **(C)** 4800 **(D)** 38880 **(E)** 43200

(A) Once Gil's seat is determined, there are 6 ways to choose the person to his right and 5 ways to choose the person to his left. After that there are 5! ways to arrange the others, so the total number of arrangements is $6 \cdot 5 \cdot 5! = 3600$.

22. How many points (x, y) satisfy the equation

$$|x^2 - 1| + |y^2 - 4| = 0?$$

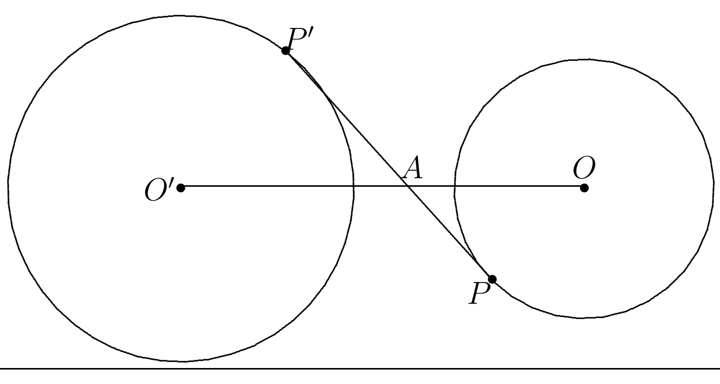
(A) 2 **(B)** 4 **(C)** 6 **(D)** 8 **(E)** infinitely many

(B) The sum of two nonnegative numbers is zero if and only if they are both zero. The only values for which $|x^2 - 1| = 0$ are $x = \pm 1$ and those for which $|y^2 - 4| = 0$ are $y = \pm 2$. Hence there are four ordered pair solutions:$(1, 2), (1, -2), (-1, 2), (-1, -2)$.

23. Two circles with centers O and O' have radii of 9 inches and 12 inches, respectively. The centers are 28 inches apart. How far from the center of circle O is the intersection of the line joining the centers with the common internal tangent PP'?

(A) 9 in. **(B)** 10 in. **(C)** 11 in. **(D)** 12 in. **(E)** 13 in.

(D) Let A denote the point of intersection of OO' and PP' and let $x = OA$. Using similar triangles we can write $\frac{x}{28-x} = \frac{9}{12}$ from which it follows that $x = 12$.



24. Suppose a and b are positive integers neither of which is a multiple of 3. Then the remainder when $a^2 + b^2$ is divided by 3

(A) must be 0 **(B)** must be 1 **(C)** must be 2

(D) may be 1 or 2 but not 0 **(E)** may be $0, 1$ or 2

(C) Since $(3n+1)^2 = 9n^2 + 6n + 1$ and $(3n+2)^2 = 9n^2 + 12n + 4$ are both one bigger than a multiple of 3, the sum $a^2 + b^2$ must be two bigger than a multiple 3.

25. For a tetrahedron $ABCD$, a plane P is called a *middle plane* if all four distances from the vertices $A, B, C,$ and D to the plane P are the same. How many middle planes are there for a given tetrahedron?

(A) 1 **(B)** 3 **(C)** 4 **(D)** 6 **(E)** 7

(E) A plane subdivides three-space into two half-spaces. There are just two cases to consider:
1) Three vertices of tetrahedron lie in one half-space and the fourth one lies in the other half-space. There are 4 middle planes of this type, one for each of the four faces. The middle planes go through middle points of edges with common vertex.
2) Two vertices of tetrahedron line in one half-space and two other vertices lie in the other half-space. There exist 3 such middle planes because the tetrahedron has 6 edges. Each middle plane goes through the middle lines of two faces which are parallel to the common edge of these faces.

26. A *snickel* is a bug which crawls among the lattice points (points with only integer coordinates) of the plane. Each move of a snickel is eight units horizontally or vertically followed by three units in a perpendicular direction. For example, from $(0,0)$ the snickel could move to any of the eight locations $(\pm 8, \pm 3), (\pm 3, \pm 8)$. What is the least number of moves required to get from $(0,0)$ to $(19,0)$?

(A) 7 **(B)** 9 **(C)** 11 **(D)** 13 **(E)** no such sequence of moves exists

(C) Let a represent the number of moves of type $(3,8)$ (a negative a refers to moves of type $(-3,-8)$, etc.), b, the number of moves of type $(8,3)$; c, the number of moves of type $(3,-8)$; and d, the number of moves of type $(8,-3)$. Then

$$a(3,8) + b(8,3) + c(3,-8) + d(8,-3) = (19,0)$$

for any path from $(0,0)$ to $(19,0)$. Check to see that $a = 2, b = -3, c = -1$ and $d = 5$ works. Thus, 11 moves will suffice. On the other hand, we must have

$$3(a+c) + 8(b+d) = 19$$

and

$$8(a-c) + 3(b-d) = 0.$$

But $a+c$ is odd $\Rightarrow a-c$ is odd $\Rightarrow a-c \neq 0$
$\Rightarrow b-d \neq 0 \Rightarrow |b-d| \geq 8$ and $|a-c| \geq 3$ so that no sequence with fewer than 11 moves can work. The sequence $(0,0), (3,8), (11,5), (19,2), (16,10), (8,7), (0,4), (8,1), (11,9), (19,6), (11,3), (19,0)$ is an example of an 11 move sequence that works.