UNIVERSITY OF NORTH CAROLINA CHARLOTTE
1998 HIGH SCHOOL MATHEMATICS CONTEST
March 9, 1998
with Solutions

1. Find the sum
$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{10 \cdot 11}.$$

(A) 1 **(B)** 2 **(C)** $\frac{10}{11}$ **(D)** $\frac{11}{12}$ **(E)** none of **A**, **B**, **C** or **D**

(C) Each summand sum can be split into the difference of two unit fractions: $\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\cdots+\dfrac{1}{10\cdot 11}=$

$$\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{10}-\frac{1}{11}\right) \quad = \quad 1-\frac{1}{11} \quad = \quad \frac{10}{11}.$$

Alternatively, use your calculator.

2. The mean test score in a math class with 27 students was 72. A student who scored 85 was moved to another class. What was the mean score of the remaining 26 students?

   (A) 69.5 **(B)** 70 **(C)** 70.5 **(D)** 71 **(E)** 71.5

   (E) The sum of the 27 scores was $27\cdot 72=1944$, so the new mean must be $(1944-85)/26=71.5$.

3. Suppose the value of a new car declines linearly over a ten year period from the original value of \$20,000 to the value \$2,000. What is the value of the car after six years?

   (A) $\$8,800$ **(B)** $\$9,200$ **(C)** $\$11,000$
   (D) $\$12,800$ **(E)** $\$13,200$

   (B) The value of the car decreases by \$1800 each year, so the value after 6 years is $20,000-6\cdot 1800=\$9200$.

4. Which of the five fractions has the smallest value?

   (A) $\dfrac{250{,}38\mathbf{6}{,}765{,}412}{250{,}38\mathbf{4}{,}765{,}412}$ **(B)** $\dfrac{250{,}38\mathbf{4}{,}765{,}412}{250{,}38\mathbf{3}{,}765{,}412}$ **(C)** $\dfrac{250{,}38\mathbf{5}{,}765{,}412}{250{,}38\mathbf{4}{,}765{,}412}$
   (D) $\dfrac{250{,}38\mathbf{6}{,}765{,}412}{250{,}38\mathbf{5}{,}765{,}412}$ **(E)** $\dfrac{250{,}38\mathbf{7}{,}765{,}412}{250{,}38\mathbf{6}{,}765{,}412}$

   (E) Option E is the smallest value:

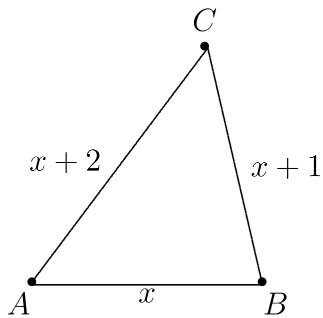
$$\begin{aligned}
A &= \frac{250{,}38\mathbf{6}{,}765{,}412}{250{,}38\mathbf{4}{,}765{,}412} &= 1+\frac{2{,}000{,}000}{250{,}384{,}765{,}412}\\
B &= \frac{250{,}38\mathbf{4}{,}765{,}412}{250{,}38\mathbf{3}{,}765{,}412} &= 1+\frac{1{,}000{,}000}{250{,}383{,}765{,}412}\\
C &= \frac{250{,}38\mathbf{5}{,}765{,}412}{250{,}38\mathbf{4}{,}765{,}412} &= 1+\frac{1{,}000{,}000}{250{,}384{,}765{,}412}\\
D &= \frac{250{,}38\mathbf{6}{,}765{,}412}{250{,}38\mathbf{5}{,}765{,}412} &= 1+\frac{1{,}000{,}000}{250{,}385{,}765{,}412}\\
E &= \frac{250{,}38\mathbf{7}{,}765{,}412}{250{,}38\mathbf{6}{,}765{,}412} &= 1+\frac{1{,}000{,}000}{250{,}386{,}765{,}412}.
\end{aligned}$$

7. In triangle ABC, $AB = x$, $BC = x + 1$, and $AC = x + 2$. Which of the following **must** be true?

   i. $x \geq 1$

   ii. $x \leq 5\sqrt{2}$

   iii. $\angle C \leq 60°$



(A) i only **(B)** ii only **(C)** iii only

(D) i and ii only **(E)** i and iii only

(E) Because $AB + BC \geq AC$, it follows that $2x + 1 \geq x + 2$, so i. is true. For any $x \geq 1$, there is a unique triangle with the given lengths, so x could be larger than $5\sqrt{2}$. Hence ii. need not be true. Since

$$\frac{\sin C}{x} = \frac{\sin A}{x+1} = \frac{\sin B}{x+2},$$

angle C is opposite the smallest side, so it is smaller than angles A and B. Because the sum of the three angles is $180°$, it follows that $3C \leq 180°$, hence iii. is true.

8. Each of the cards shown below has a number on one side and a letter on the other. How many of the cards must be turned over to prove the correctness of the statement:

   Every card with a vowel on one side has a prime number on the other side.

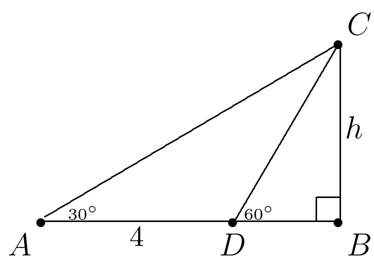
$\boxed{A}$ $\boxed{B}$ $\boxed{E}$ $\boxed{4}$ $\boxed{5}$ $\boxed{6}$ $\boxed{8}$

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

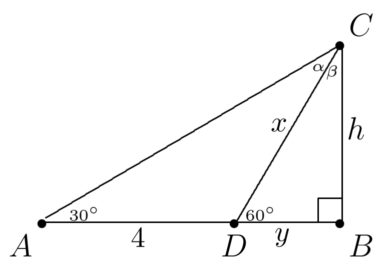
(E) We must turn over every card with a non-prime number (to be sure it doesn't have a vowel on the other side), and every card with a vowel (to be sure it has a prime on the other side). Hence we must overturn five cards.

9. In right triangle ABC, the point D on $\overline{AB}$ is 4 units from A, $\angle CDB = 60°$ and $\angle CAB = 30°$. What is the altitude h?

(A) 3 **(B)** $2\sqrt{3}$ **(C)** $\sqrt{14}$ **(D)** 4 **(E)** $3\sqrt{2}$



(B) Let $x = DC, y = DB, \alpha = \angle ACD$, and $\beta = \angle DCB$. It is easy to see that $\alpha = \beta = 30°$, so $x = 4$ and $y = \frac{1}{2}x = 2$. Then $h^2 = x^2 - y^2 = 4^2 - 2^2 = 12$, so $h = 2\sqrt{3}$.



10. The numbers x, y, and z satisfy

$$|x+2|+|y+3|+|z-5|=1.$$

Which of the following could be $|x+y+z|$?

(A) 0 **(B)** 2 **(C)** 5 **(D)** 7 **(E)** 10

(A) By the triangle inequality, note that

$$|x+y+z|=|x+2+y+3+z-5|\leq|x+2|+|y+3|+|z-5|=1.$$

Hence A is the only possible value. On the other hand, $x=-2.5, y=-3$ and $z=5.5$ satisfies the equation, so $|x+y+z|$ can be zero.

11. Suppose $a<0$ and $|a|\cdot x\leq a$. Evaluate $|x+1|-|x-2|$.

(A) -3 **(B)** -1 **(C)** $-2x+1$ **(D)** $2x+3$ **(E)** 3

(A) Divide both sides of the second inequality by $|a|$ to obtain $x\leq-1$. In this case $|x+1|=-(x+1)$ and $|x-2|=2-x$, so $|x+1|-|x-2|=-(x+1)-(2-x)=-3$.

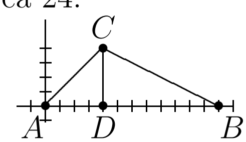
12. The set of points satisfying the three inequalities

$$y\geq 0, \quad y\leq x, \text{ and } \quad y\leq 6-x/2$$

is a triangular region with an area of

(A) 12 **(B)** 18 **(C)** 24 **(D)** 36 **(E)** 48

(C) The vertices of the triangle can be obtained by solving the *equations* simultaneously in pairs: $(0,0),(12,0)$, and $(4,4)$. The triangle has base 12 and altitude 4, hence area 24.

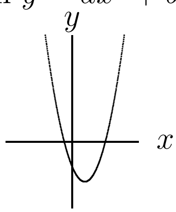


13. If $x > 5$, which of the following is smallest?

(A) $\dfrac{5}{x}$ **(B)** $\dfrac{5}{x-1}$ **(C)** $\dfrac{x}{5}$ **(D)** $\dfrac{5}{x+1}$ **(E)** $\dfrac{x+1}{5}$

(D) The fractions $\frac{x}{5}$ and $\frac{x+1}{5}$ are both larger than 1. The other three are less than 1 and have the same numerator, so the smallest is the one with the largest denominator.

14. The graph of the quadratic function $y = ax^2 + bx + c$ is shown below.



Which of the following is true?

(A) $ac < 0$ and $ab < 0$ **(B)** $ac < 0$ and $ab > 0$

(C) $ac > 0$ and $ab < 0$ **(D)** $ac > 0$ and $ab > 0$

(E) At least one of a, b, and c could be zero.

(A) The y-intercept c is negative, and the constant a is positive since the parabola opens upward. The x-coordinate of the vertex which is visibly positive, is given by the formula $x = -b/2a$. Since $a > 0$, the number $-b/2a$ is positive only if $b < 0$. Thus, $ab < 0$ and $ac < 0$.

15. A drawer contains exactly six socks–two are green, two are red, two are blue. If two socks are selected at random without replacement, what is the probabilty that they match?

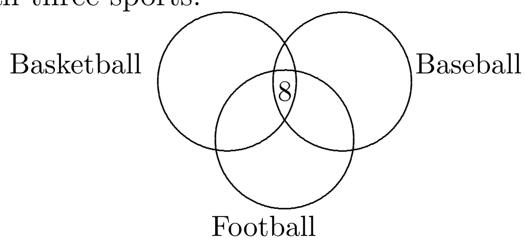
(A) $\dfrac{1}{6}$ **(B)** $\dfrac{1}{5}$ **(C)** $\dfrac{1}{4}$ **(D)** $\dfrac{1}{3}$ **(E)** $\dfrac{1}{2}$

(B) Once the first sock has been picked, the probability that the second matches it is one-fifth.

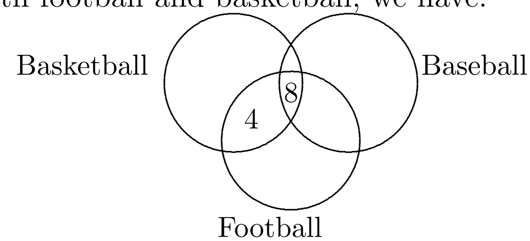
16. Of the members of three athletic teams at Harding High, 21 are on the basketball team, 26 are on the baseball team, and 29 are on the football team. A total of 14 play both baseball and basketball; 15 play both baseball and football; and 12 play football and basketball. There are eight who play all three sports. How many students play on at least one of the teams?

(A) 43 **(B)** 49 **(C)** 51 **(D)** 76 **(E)** 84

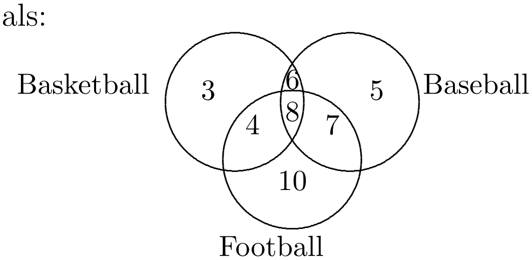
(A) Construct a Venn diagram as shown below. Start with the fact that 8 students play all three sports.



Since 12 play both football and basketball, we have:



Proceeding in this manner until all the given information is used yields the following totals:



Hence the total number of members is 43.

17. Five test scores have a mean (average score) of 91, a median (middle score) of 92 and a mode (most frequent score) of 95. The sum of the two lowest test scores is

(A) 172 **(B)** 173 **(C)** 174 **(D)** 178 **(E)** 179

(B) The five scores must be of the form $x, y, 92, 95, 95$ and have a sum of $5 \cdot 91 = 455$. Therefore $x + y = 173$.

18. In his last will, a farmer asked that his horses be distributed among his four sons. The oldest was to get one third of the herd, the second oldest, one fourth of the herd, and each of the two youngest ones was to get one fifth of the herd. When the sons read the will, they were puzzled because none of them were going to get an integer number of horses. At that moment, they discovered that a baby horse had just been born. Each son would receive an integer number of horses, but the baby horse would be left over. How many horses did the farmer have originally?

(A) 29 **(B)** 59 **(C)** 89 **(D)** 119 **(E)** 239

(B) Suppose the farmer originally had N horses. Splitting $N+1$ horses as described in the will and recalling that one horse is not distributed leads to

$$\frac{N+1}{3} + \frac{N+1}{4} + \frac{2(N+1)}{5} = N$$

Adding the fractions yields

$$\frac{59(N+1)}{60} = N$$

Hence $N = 59$.

19. If a, b, c and d are four positive numbers such that $\frac{a}{b} < \frac{c}{d}$, then

(A) $ab < dc$ **(B)** $a + c < b + d$ **(C)** $a + d < b + c$

(D) $\dfrac{a+c}{b+d} < \dfrac{c}{d}$ **(E)** $\dfrac{c-a}{d-b} < \dfrac{c}{d}$

(D) Since $\frac{a}{b} < \frac{c}{d}$, we have $ad < bc$. Add cd to both sides: $ad+cd < bc+dc$ so $d(a + c) < (b + d)c$. Dividing both sides by $d(b + d)$ yields the result. To see that the other four options fail, let $a = 5, b = c = 3$, and $d = 1$. This choice eliminates options A, B, and C while $a = 1, b = c = 2$, and $d = 3$ shows that E is not always true.

20. How many odd three-digit numbers have three digits different?

(A) 60 **(B)** 288 **(C)** 300 **(D)** 320 **(E)** 405

(D) Since the number is odd, there are 5 ways to choose the units digit. There are two cases, which depends on the tens digit.

(a) If the tens digit is 0, then 8 ways to choose the hundreds digit for a total of $8 \cdot 1 \cdot 5 = 40$ numbers of the type required.

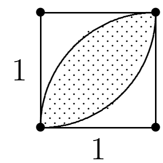
(b) If the tens digit is not 0, there are 8 choices for it. Then there are 7 ways to choose the hundreds digit for a total of $7 \cdot 8 \cdot 5 = 280$ numbers of the type required.

Thus there are $40 + 280 = 320$ such numbers altogether.

OR

Let the three digit number be $100a + 10b + c$ with digit a being one of $1, 2, 3, 4, 5, 6, 7, 8, 9$; digit b being one of $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$; digit c being one of $1, 3, 5, 7, 9$. One distinguishes four cases a, b both even: There are $4 \cdot 4 \cdot 5 = 80$ possibilities. a, b both odd: There are $5 \cdot 4 \cdot 3 = 60$ possibilities. a odd and b even: There are $5 \cdot 5 \cdot 4 = 100$ possibilities. a even and b odd: There are $4 \cdot 5 \cdot 4 = 80$ possibilities. Taken together, it adds up to $80 + 60 + 100 + 80 = 320$ possibilities.

21. What is the area of the region common to two unit circles whose centers are $\sqrt{2}$ apart?



(A) $\frac{1}{2}\left(1-\frac{\pi}{4}\right)$ **(B)** $\frac{\pi}{2}$ **(C)** $1-\frac{\pi}{4}$ **(D)** $\frac{\pi}{2}-1$ **(E)** $\frac{1}{2}$

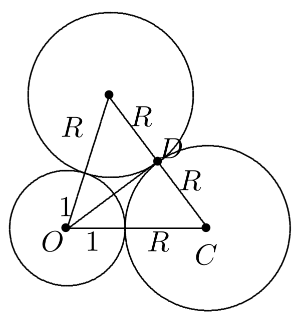
(D) Half of the unshaded area is $1-\frac{\pi}{4}$ so the shaded area is

$$1-2\left(1-\frac{\pi}{4}\right)=\frac{\pi}{2}-1.$$

22. Five circles with equal radii are situated in the plane so that each is tangent to two others and externally tangent to a unit circle. Find the radius of each of the five circles, rounded to two decimal places.

(A) 1.39 **(B)** 1.41 **(C)** 1.43 **(D)** 1.45 **(E)** 1.47

(C) The centers of the circles are the vertices of a regular pentagon. Thus, the angle between adjacent circles is $360°/5 = 72°$, and in the right triangle OCD we have $\angle COD = 36°$. Hence $\sin 36° = \frac{R}{R+1}$ from which it follows that $R \approx 1.43$.



23. If Sam and Peter are among 6 men who are seated at random in a row, the probability that exactly 2 men are seated between them is

(A) 1/10 **(B)** 1/8 **(C)** 1/5 **(D)** 1/4 **(E)** 4/15

(C) The total number of the permutations of 6 is $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$. There are $6 \cdot 4!$ outcomes with exactly two men seated between Sam and Peter. To see this note that the blanks can be filled in in four factorial ways: $S, \_, \_, P, \_, \_$ for each of the six possible positions of the S and P. The probability is therefore $6 \cdot 4!/6! = 1/5$.

OR

Note that once Peter has been seated, there is just one place out of the five remaining for Sam to sit so that the two are separated by exactly two men.

24. Four congruent triangular corners are cut off an 11×13 rectangle. The resulting octagon has eight edges of equal length. What is this length?

(A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

(C) Let x and y be the sides of the triangles on the sides of the rectangle of lengths 13 and 11. Because all sides of the octagon are equal

$$\begin{aligned} \sqrt{x^2 + y^2} &= 13 - 2x \\ 13 - 2x &= 11 - 2y \end{aligned}$$

The second equation yields $y = x - 1$. Plugging into the first equation and squaring yields

$$\begin{aligned} x^2 + (x-1)^2 &= (13 - 2x)^2 \\ 2x^2 - 2x + 1 &= 169 - 52x + 4x^2 \\ 168 - 50x + 2x^2 &= 0 \\ x^2 - 25x + 84 &= 0 \end{aligned}$$

This quadratic equation has the two solution $x = 21$ or $x = 4$. Only the last solution can occur since $13 - 2x > 0$. One gets $y = x - 1 = 3$ and finally $\sqrt{x^2 + y^2} = 5$ for the side of the octagon.

25. Evaluate the continued fraction

$$2 + \cfrac{1}{2 + \frac{1}{2+\frac{1}{2+\cdots}}}$$

(A) $\sqrt{5}$ **(B)** $\sqrt{6}$ **(C)** $1+\sqrt{2}$ **(D)** $1+\sqrt{3}$ **(E)** 3

(C) Let $x = 2 + \cfrac{1}{2 + \frac{1}{2+\frac{1}{2+\cdots}}}$. Then $2 + \frac{1}{x} = x$ from which it follows that $x^2 - 2x - 1 = 0$. Because $x > 0$, we have $x = \frac{2+\sqrt{4-4(-1)}}{2} = 1 + \sqrt{2}$.

26. Let x and b be positive integers. Suppose that x is represented as 324 in base b, and x is represented as 155 in base $b+2$. What is b?

(A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

(A) Note that

$$\begin{aligned} 3b^2 + 2b + 4 &= 1(b+2)^2 + 5(b+2) + 5 \\ &= b^2 + 4b + 4 + 5b + 10 + 5 \\ &= b^2 + 9b + 19, \end{aligned}$$

so $2b^2 - 7b - 15 = 0$. Thus, $(b-5)(2b+3) = 0$. Only $b = 5$ makes sense.

27. It is known that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. What is the value of

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}?$$

(A) $\frac{\pi^2}{36}$ **(B)** $\frac{\pi^2}{12}$ **(C)** $\frac{\pi^2}{8}$ **(D)** $\frac{\pi^2}{7}$ **(E)** $\frac{2\pi^2}{9}$

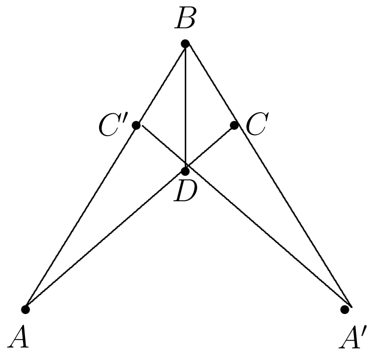
(C) Notice that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} + \sum_{n=1}^{\infty} \frac{1}{(2n)^2}.$$

Therefore,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} &= \sum_{n=1}^{\infty} \frac{1}{n^2} - \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2} \\ &= \frac{3}{4} \sum_{n=1}^{\infty} \frac{1}{n^2} \\ &= \frac{3}{4} \left(\frac{\pi^2}{6} \right) \\ &= \frac{\pi^2}{8} \end{aligned}$$

28. In triangle $ABC, AB = 20, BC = 5$, and $\angle ABC = 60°$. The triangle is reflected in the plane about the bisector of angle ABC to produce a new triangle $A'BC'$ as shown. What is the area of the region enclosed by the union of the two triangles?



(A) $20\sqrt{3}$ **(B)** $24\sqrt{3}$ **(C)** $40\sqrt{2}$ **(D)** $40\sqrt{3}$ **(E)** $50\sqrt{2}$

(D) Let the reflection of points A and C be A' and C' respectively. Label point D such that $\overline{BD}$ is the bisector of angle ABC and D is on $\overline{AC}$. Since angles $ABD, A'BD, CBD,$ and $C'BD$ are all congruent, the reflection places C' on the segment $\overline{AB}$ and A' on the extension of $\overline{BC}$. The area of $\triangle ABC$ is found to be $[ABC] = \frac{1}{2}AB \cdot BC \sin\angle ABC = \frac{1}{2} \cdot 20 \cdot 5 \cdot \sin 60^\circ = 25\sqrt{3}$. The requested area is twice the area of triangle ABD since $\overline{BD}$ lies on the line of reflection. Since $\overline{BD}$ is an angle bisector, we have

$$\begin{aligned}\frac{AD}{DC} = \frac{AB}{BC} &= \frac{20}{5} = 4 \text{ and}\\ 2[ABD] = 2 \cdot (AD/AC) \cdot [ABC] &= 2 \cdot (4/5) \cdot [ABC]\\ = (8/5) \cdot 25\sqrt{3} &= 40\sqrt{3}.\end{aligned}$$

OR

Apply the law of cosines to $\triangle ABC$ and obtain

$$x + y = \sqrt{20^2 + 5^2 - 2 \cdot 20 \cdot 5 \cdot \cos 60^\circ} = 5\sqrt{13}.$$

Apply the law of sines to triangles BCD and BAD to get

$$\frac{5}{\sin\theta} = \frac{x}{\sin 30^\circ}$$

and

$$\frac{20}{\sin(180 - \theta)} = \frac{y}{\sin 30^\circ}.$$

Thus $x/y = 5/20$ and $y = \frac{4}{5}(x + y) = 4\sqrt{13}$. Applying the law of sines to $\triangle ABC$ yields

$$\frac{5}{\sin\alpha} = \frac{x + y}{\sin 60^\circ}$$

which implies that $\sin\alpha = \frac{\sqrt{3}}{2\sqrt{13}}$. Thus the desired area is $2 \cdot [ABD] = 2 \cdot \frac{1}{2} \cdot 20h = 20 \cdot y \sin\alpha = 40\sqrt{3}$.