PROBLEM DEPARTMENT

ASHLEY AHLIN AND HAROLD REITER*

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

All correspondence should be addressed to Harold Reiter, Department of Mathematics, University of North Carolina Charlotte, 9201 University City Boulevard, Charlotte, NC 28223-0001 or sent by email to `hbreiter@uncc.edu`. *Electronic submissions using LaTeX are encouraged. Other electronic submissions are also encouraged. Please submit each proposal and solution preferably typed or clearly written on a separate sheet (one side only) properly identified with name, affiliation, and address. Solutions to problems in this issue should be mailed to arrive by March 1, 2010. Solutions identified as by students are given preference.*

Problems for Solution.

1204. *Proposed by Sam Vandervelde, St. Lawrence University, Canton, NY*

Prove that there exists a real number α satisfying $\lfloor \alpha^n \rfloor \equiv n \pmod 5$ for all $n \in \mathbb{N}$.

1205. *Proposed by Peter A. Lindstrom, Batavia, NY*

The concatenation of positive integers p and q, denoted $p \,||\, q$, is defined as

$$p \,||\, q = p \cdot \left(10^{\lfloor \log_{10} q \rfloor + 1}\right) + q,$$

where $\lfloor x \rfloor$ is the floor function. For example $37 \,||\, 21 = 37(10^{\lfloor \log_{10} 21 \rfloor + 1} + 21 = 37(10^{1+1}) + 21 = 3721$.

Find infinitely many triplets of positive integers a, b, c such that

1. $(a \,||\, b) \,||\, c$ is not a palindrome, and
2. $(a^2 \,||\, b^2) \,||\, c^2$ is a palindrome.

1206. *Proposed by Stas Molchanov, University of North Carolina Charlotte*

A square is said to be inscribed in a quadrilateral if each vertex of the square belongs to a different edge of the quadrilateral. Find necessary and sufficient condition on a parallelogram in order to have an inscribed square.

1207. *Proposed by R. M. Welukar, K. S. Bhanu, and M. N. Deshpande, Open University and Institute of Science, India.*

Let $F_k, F_{k+1}, \ldots F_{k+4n-1}$ be arranged in a $2 \times 2n$ matrix as shown below.

$$\begin{matrix} 1 & 2 & 3 & 4 & \cdots & 2n \end{matrix}$$
$$\begin{pmatrix} F_k & F_{k+3} & F_{k+4} & F_{k+7} & \cdots & F_{k+4n-1} \\ F_{k+1} & F_{k+2} & F_{k+5} & F_{k+6} & \cdots & F_{k+4n-2} \end{pmatrix}$$

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Show that the sum of the elements of the first and second rows denoted by R_1 and R_2 respectively can be expressed as

$$R_1 = 2F_{2n}F_{2n+k}$$
$$R_2 = F_{2n}F_{2n+k+1}$$

where $\{F_n, n \geq 1\}$ denotes the Fibonacci sequence.

1208. *Proposed by Matthew McMullen, Otterbein College, Westerville, OH.*

Let k be a positive integer with $k \equiv 1 \pmod 4$. Define x_k to be the solution to $\tan x = x$ on the interval $\left[\frac{(k-1)\pi}{2}, \frac{k\pi}{2}\right)$. Let $A = A(k) > 0$ be chosen so that the equation $\sin x = Ax$ has exactly k solutions. Show that A is unique and that $x_k < 1/A < k\pi/2$.

1209. *Proposed by José Luis Díaz-Barrero, Universidad Politécnica de Cataluña, Barcelona, Spain.*

Let R and r denote the radii of the circumcircle and the incircle of a triangle ABC with sides a, b, c and semi-perimeter s. Prove that

$$\frac{(s-a)^4}{c(s-b)} + \frac{(s-b)^4}{a(s-c)} + \frac{(s-c)^4}{b(s-a)} \geq \frac{3r}{4}\sqrt[3]{\frac{Rs^2}{2}}$$

1210. *Proposed by Robert Gebhardt, Hopatcong, NJ.*

Find an equation for the plane tangent to the ellipsoid $a^2x^2 + b^2y^2 + c^2z^2 = d^2$ in the first octant such that the volume in the first octant bounded by this plane and the coordinate planes is minimum.

1211. *Proposed by Greg Oman, Otterbein College, Westerville, OH.*

Let $\sum_{n=1}^{\infty} a_n$ be a convergent series with positive terms. Prove that $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}$ converges. *Note from the poser:* Since this problem is well-known, the challenge is to provide an elementary solution, one that does not invoke the Cauchy-Schwartz Inequality.

1212. *Proposed by Arthur Holshouser, Charlotte, NC.*

Is it possible to paint the 750 faces of 125 unit cubes using five colors, say red, white, blue, green and yellow, in such a way that for each color c, a $5 \times 5 \times 5$ cube can be built whose entire surface is color c.

1213. *Proposed by Scott D. Kominers, student, Harvard University, Cambridge, MA. and Paul M. Kominers, student, MIT, Cambridge, MA.*

In Problem 1176 of this *Journal*, readers were challenged to determine for which nonzero ρ there exists a real (or complex) sequence $\{a_1, a_2, a_3, \ldots\}$ such that

$$\rho = \frac{a_1 + \cdots + a_n}{a_{n+1} + \cdots + a_{2n}} \tag{1}$$

for all $n > 0$.

Here, we ask a similar question: For which nonzero ρ does there exist an *integer* sequence $\{a_1, a_2, a_3, \ldots\}$ such that (1) holds for all $n > 0$?