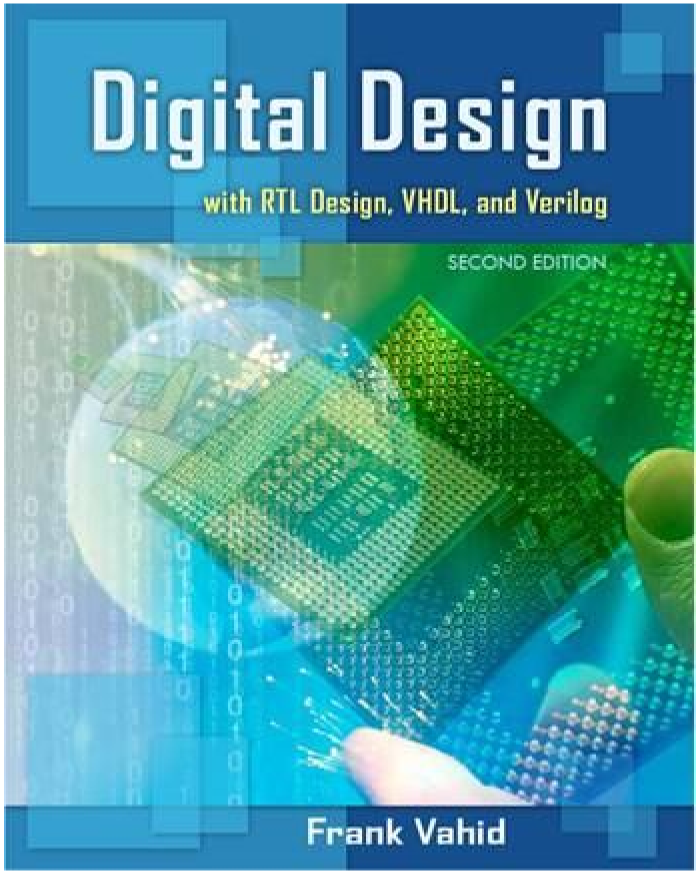


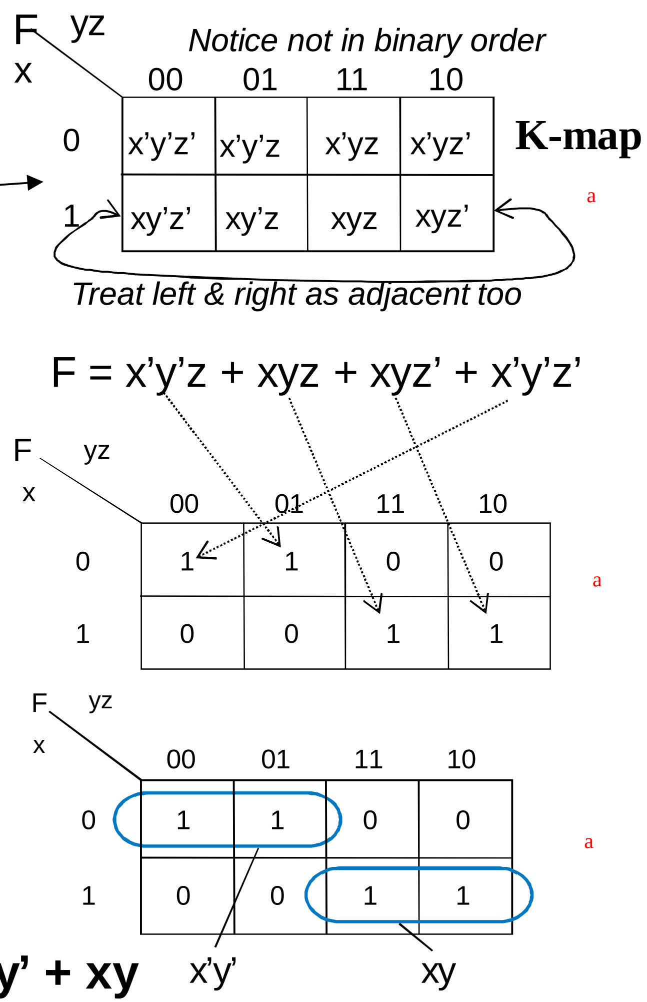
Digital Design

Chapter 6:
Optimizations and Tradeoffs

Slides to accompany the textbook *Digital Design, with RTL Design, VHDL, and Verilog*, 2nd Edition,
by Frank Vahid, John Wiley and Sons Publishers, 2010.
http://www.ddvahid.com

Karnaugh Maps for Two-Level Size Optimization

- Easy to miss possible opportunities to combine terms when doing algebraically
- ***Karnaugh Maps (K-maps)***
  - **Graphical** method to help us find opportunities to combine terms
  - Minterms differing in one variable are *adjacent* in the map
  - Can clearly see opportunities to combine terms – look for adjacent 1s
    - For F, clearly two opportunities
    - Top left circle is *shorthand* for: x'y'z'+x'y'z = x'y'(z'+z) = x'y'(1) = x'y'
    - Draw circle, write term that has all the literals except the one that changes in the circle
      - Circle xy, x=1 & y=1 in both cells of the circle, but z changes (z=1 in one cell, 0 in the other)
    - Minimized function: OR the final terms



K-map

Notice not in binary order

F x \ yz	00	01	11	10
0	x'y'z'	x'y'z	x'yz	x'yz'
1	xy'z'	xy'z	xyz	xyz'

Treat left & right as adjacent too

F = x'y'z + xyz + xyz' + x'y'z'

F x \ yz	00	01	11	10
0	1	1	0	0
1	0	0	1	1

F x \ yz	00	01	11	10
0	1	1	0	0
1	0	0	1	1

F = x'y' + xy

Easier than algebraically:

F = xyz + xyz' + x'y'z' + x'y'z
F = xy(z + z') + x'y'(z + z')
F = xy*1 + x'y'*1
F = xy + x'y'

K-maps

- Four adjacent 1s means two variables can be eliminated
  - Makes intuitive sense – those two variables appear in all combinations, so one term *must* be true
  - Draw one big circle – *shorthand* for the algebraic transformations above

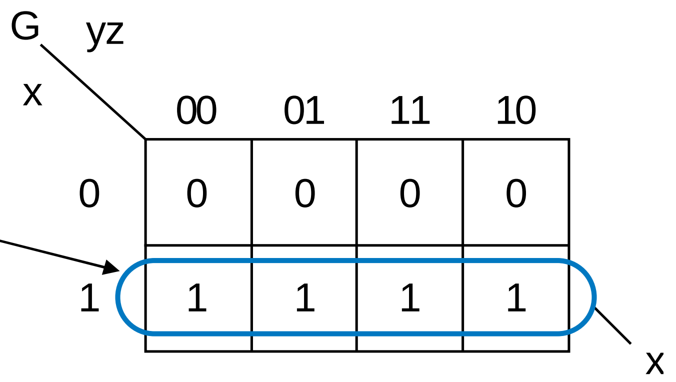
$G = xy'z' + xy'z + xyz + xyz'$

$G = x(y'z'+ y'z + yz + yz')$ *(must be true)*

$G = x(y'(z'+z) + y(z+z'))$

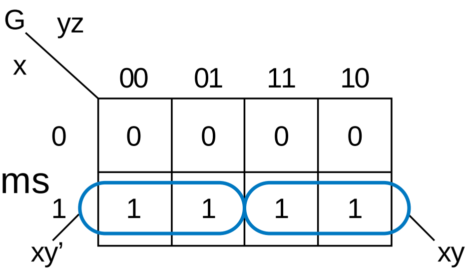
$G = x(y'+y)$

$G = x$



G (x \ yz)	00	01	11	10
0	0	0	0	0
1	1	1	1	1

Draw the biggest circle possible, or you'll have more terms than really needed



G (x \ yz)	00	01	11	10
0	0	0	0	0
1	1	1	1	1

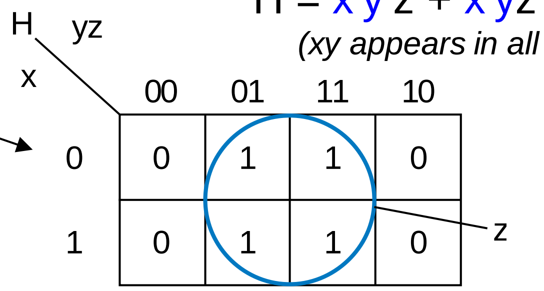
a

K-maps

- Four adjacent cells can be in shape of a square
- *OK* to cover a 1 twice
  - Just like duplicating a term
    - Remember, c + d = c + d + d
- No *need* to cover 1s more than once
  - Yields extra terms – not minimized

H = x’y’z + x’yz + xy’z + xyz

(xy appears in all combinations)



	00	01	11	10
0	0	1	1	0
1	0	1	1	0

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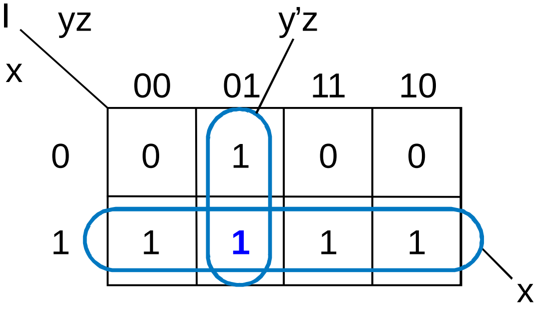

	00	01	11	10
0	0	1	0	0
1	1	**1**	1	1

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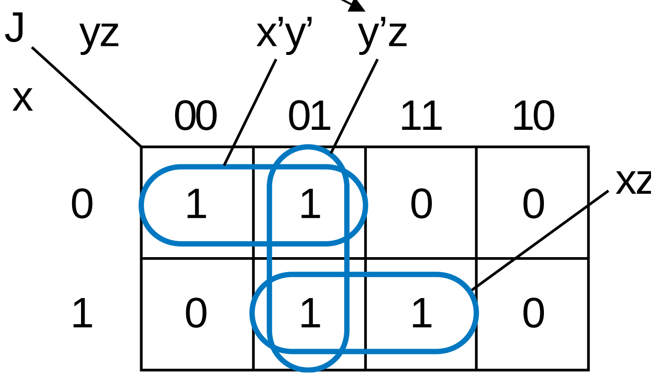
The two circles are shorthand for:

I = x’y’z + xy’z’ + xy’z + xyz + xyz’

I = x’y’z + xy’z + xy’z’ + xy’z + xyz + xyz’

I = (x’y’z + xy’z) + (xy’z’ + xy’z + xyz + xyz’)

I = (y’z) + (x)



	00	01	11	10
0	1	1	0	0
1	0	1	1	0

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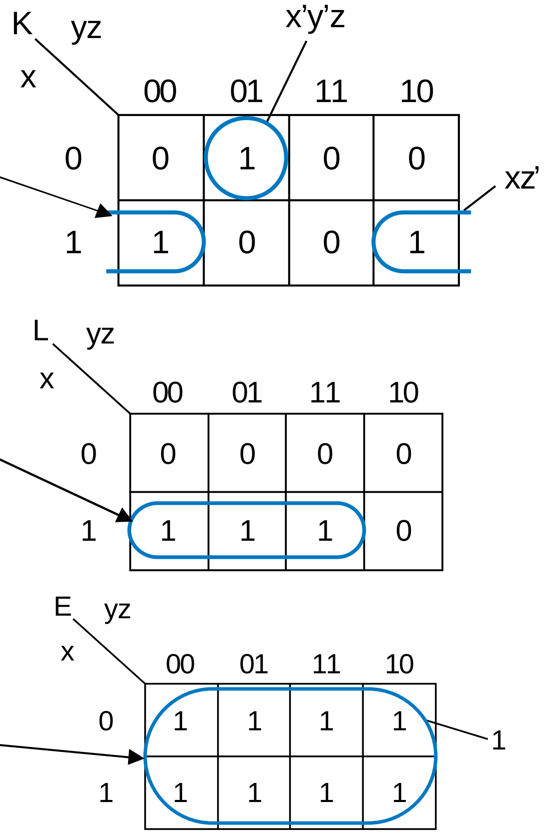
K-maps

- Circles can cross left/right sides
  - Remember, edges are adjacent
    - Minterms differ in one variable only
- Circles must have 1, 2, 4, or 8 cells – 3, 5, or 7 not allowed
  - 3/5/7 doesn't correspond to algebraic transformations that combine terms to eliminate a variable
- Circling all the cells is OK
  - Function just equals 1



x \ yz	00	01	11	10
0	0	1	0	0
1	1	0	0	1

x \ yz	00	01	11	10
0	0	0	0	0
1	1	1	1	0

x \ yz	00	01	11	10
0	1	1	1	1
1	1	1	1	1

a

a

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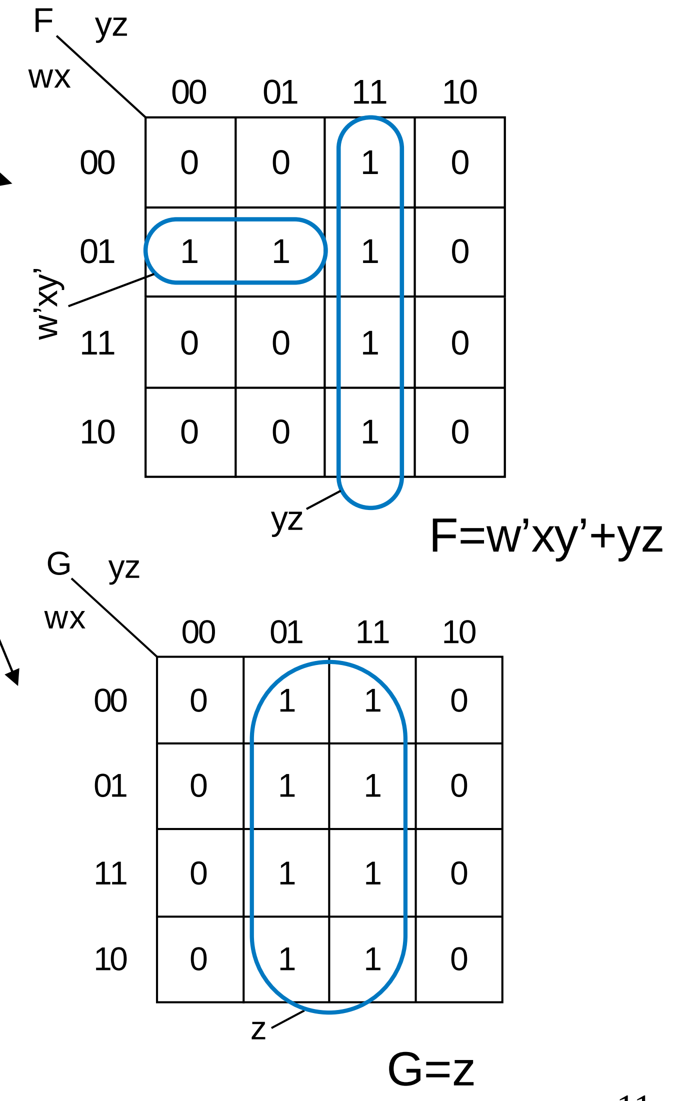
K-maps for Four Variables

- Four-variable K-map follows same principle
  - Adjacent cells differ in one variable
  - Left/right adjacent
  - Top/bottom also adjacent
- 5 and 6 variable maps exist
  - But hard to use
- Two-variable maps exist
  - But not very useful – easy to do algebraically by hand

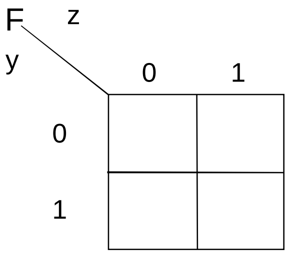


F wx \ yz	00	01	11	10
00	0	0	1	0
01	1	1	1	0
11	0	0	1	0
10	0	0	1	0

F=w'xy'+yz

G wx \ yz	00	01	11	10
00	0	1	1	0
01	0	1	1	0
11	0	1	1	0
10	0	1	1	0

G=z



Two-Level Size Optimization Using K-maps

General K-map method

1. *Convert* the function's equation into sum-of-minterms form
2. *Place* 1s in the appropriate K-map cells for each minterm
3. *Cover* all 1s by drawing the fewest largest circles, with every 1 included at least once; write the corresponding term for each circle
4. *OR* all the resulting terms to create the minimized function.

Two-Level Size Optimization Using K-maps

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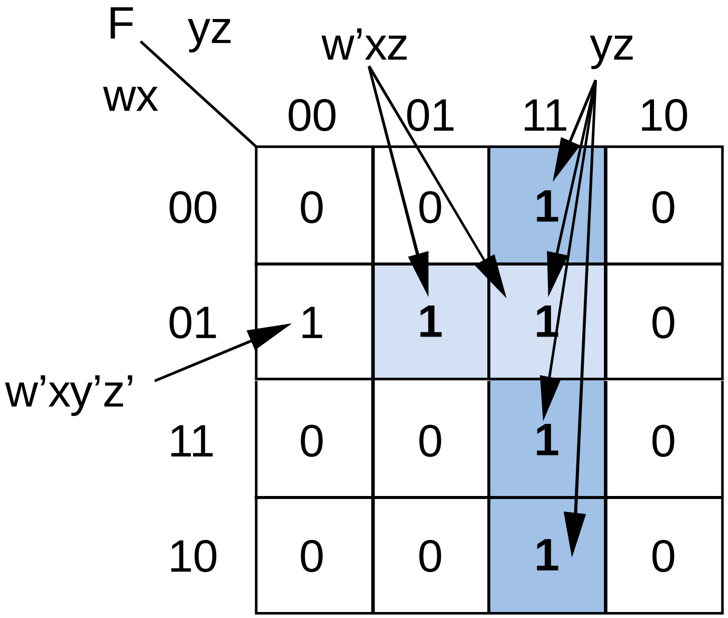
1. *Convert* the function’s equation into sum-of-minterms form
2. *Place* 1s in the appropriate K-map cells for each minterm

Common to revise (1) and (2):

- Create *sum-of-products*
- Draw 1s for each product

Ex: F = w'xz + yz + w'xy'z'



F wx \ yz	00	01	11	10
00	0	0	**1**	0
01	1	**1**	**1**	0
11	0	0	**1**	0
10	0	0	**1**	0

Two-Level Size Optimization Using K-maps

General K-map method

1. *Convert* the function's equation into sum-of-minterms form
2. *Place* 1s in the appropriate K-map cells for each minterm
3. *Cover* all 1s by drawing the fewest largest circles, with every 1 included at least once; write the corresponding term for each circle
4. *OR* all the resulting terms to create the minimized function.

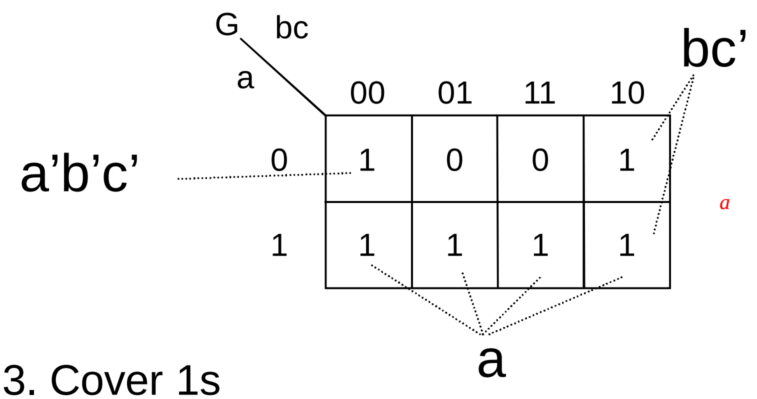
Example: Minimize:

G = a + a'b'c' + b*(c' + bc')

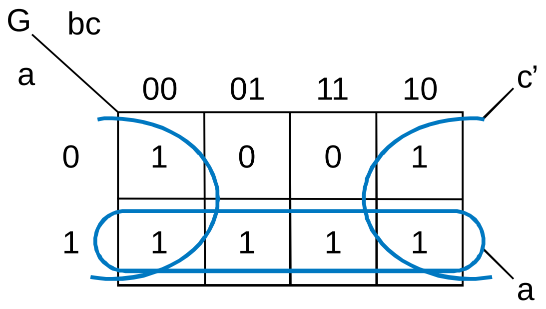
1. Convert to sum-of-products

G = a + a'b'c' + bc' + bc'

2. Place 1s in appropriate cells



3. Cover 1s



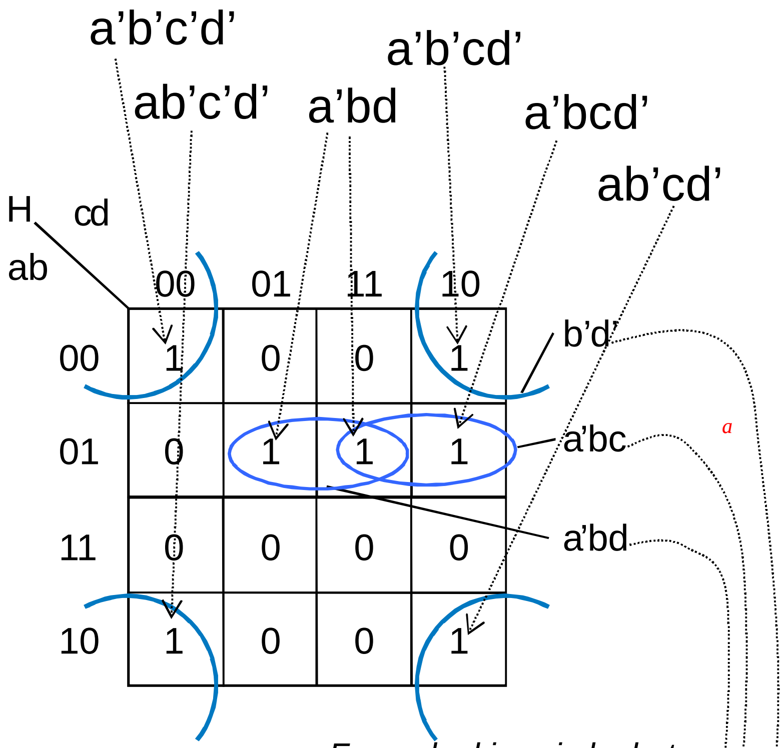
4. OR terms: **G = a + c'**

Two-Level Size Optimization Using K-maps

– Four Variable Example

- Minimize:
  - H = a’b’(cd’ + c’d’) + ab’c’d’ + ab’cd’ + a’bd + a’bcd’

1. Convert to sum-of-products:
   - H = a’b’cd’ + a’b’c’d’ + ab’c’d’ + ab’cd’ + a’bd + a’bcd’
2. Place 1s in K-map cells
3. Cover 1s
4. OR resulting terms



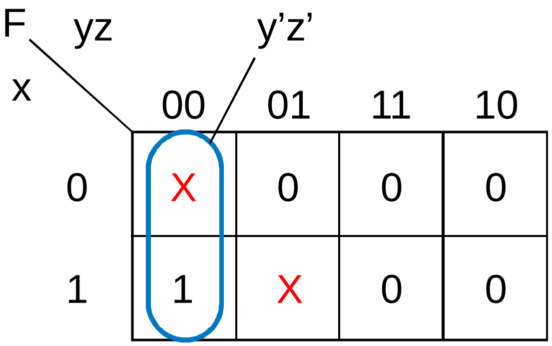
ab \ cd	00	01	11	10
00	1	0	0	1
01	0	1	1	1
11	0	0	0	0
10	1	0	0	1

Funny-looking circle, but remember that left/right adjacent, and top/bottom adjacent

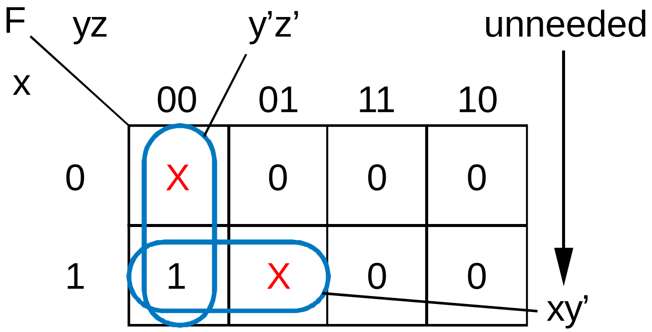
H = b’d’ + a’bc + a’bd

Don’t Care Input Combinations

- What if we know that particular input combinations can never occur?
  - e.g., Minimize F = xy’z’, given that x’y’z’ (xyz=000) can *never* be true, and that xy’z (xyz=101) can *never* be true
  - So it doesn’t matter what F outputs when x’y’z’ or xy’z is true, because those cases *will never occur*
  - Thus, make F be 1 or 0 for those cases *in a way that best minimizes the equation*
- On K-map
  - Draw **X**s for don’t care combinations
    - Include X in circle ONLY if minimizes equation
    - Don’t include other Xs



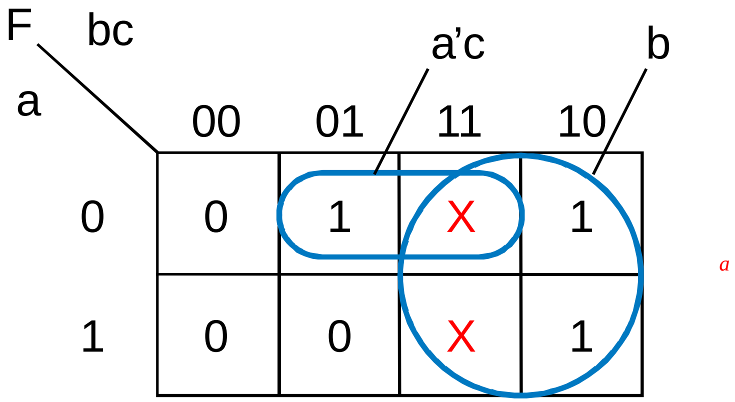
Good use of don’t cares



Unnecessary use of don’t cares; results in extra term

Optimization Example using Don’t Cares

- Minimize:
  - F = a’bc’ + abc’ + a’b’c
  - Given don’t cares: a’bc, abc



a \ bc	00	01	11	10
0	0	1	X	1
1	0	0	X	1

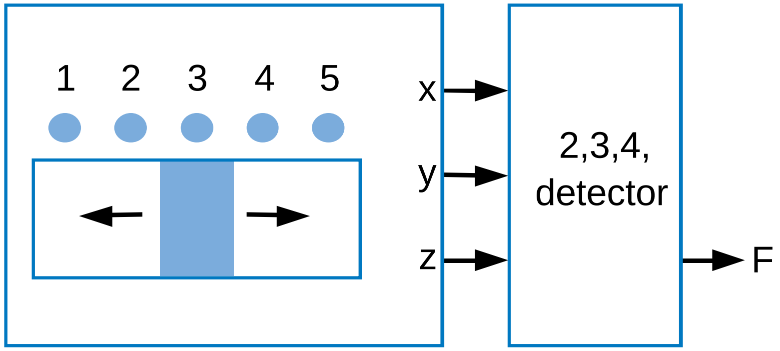
F = a’c + b

- Note: Introduce don’t cares with caution
  - Must be *sure* that we really don’t care what the function outputs for that input combination
  - If we do care, even the slightest, then it’s probably safer to set the output to 0

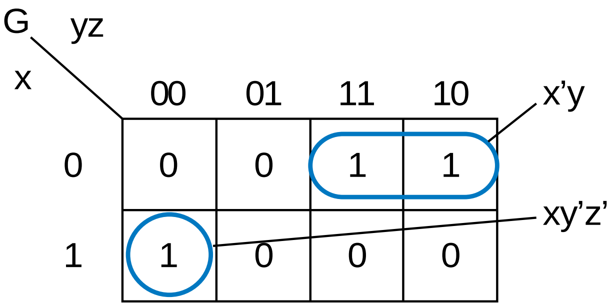
Optimization with Don’t Cares Example:

Sliding Switch

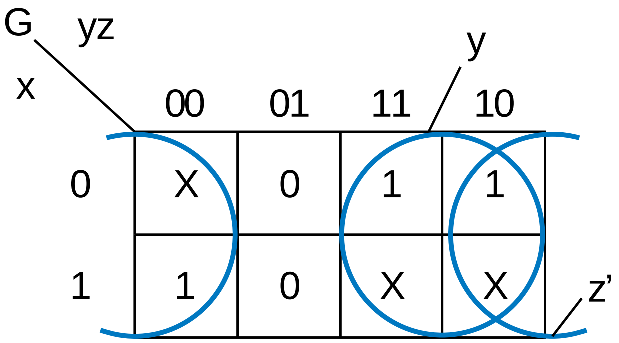
- Switch with 5 positions
  - 3-bit value gives position in binary
- Want circuit that
  - Outputs 1 when switch is in position 2, 3, or 4
  - Outputs 0 when switch is in position 1 or 5
  - Note that the 3-bit input can never output binary 0, 6, or 7
    - Treat as don’t care input combinations



Without don’t cares:
F = x’y + xy’z’



x \ yz	00	01	11	10
0	0	0	1	1
1	1	0	0	0

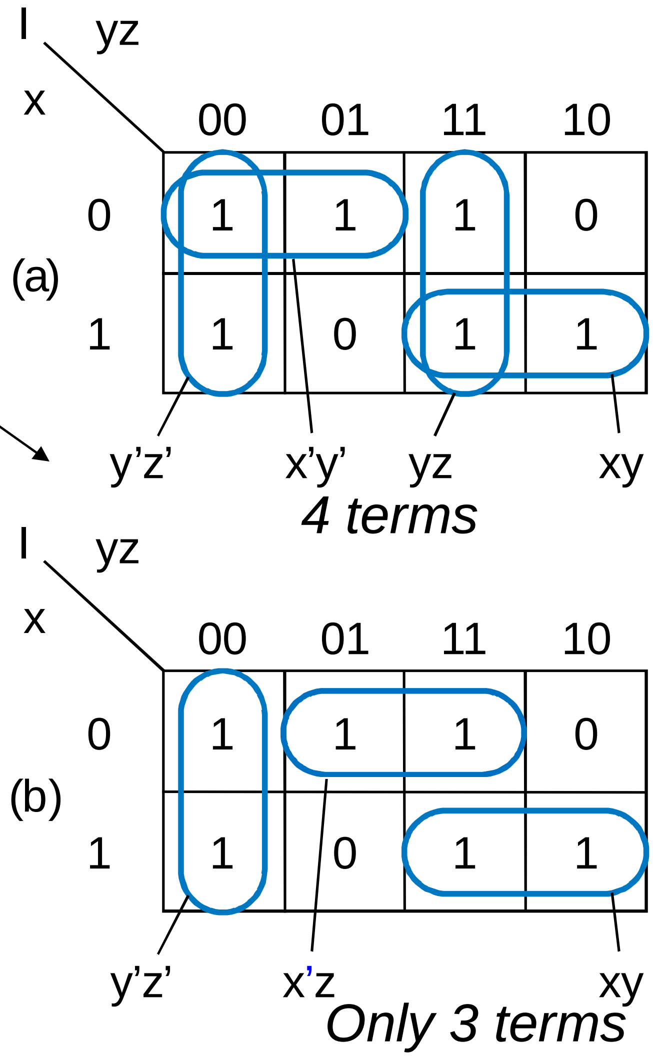


x \ yz	00	01	11	10
0	X	0	1	1
1	1	0	X	X

With don’t cares:
F = y + z’

Automating Two-Level Logic Size Optimization

- Minimizing by hand
  - Is hard for functions with 5 or more variables
  - May not yield minimum cover depending on order we choose
  - Is error prone
- Minimization thus typically done by automated tools
  - ***Exact algorithm***: finds optimal solution
  - ***Heuristic***: finds good solution, but not necessarily optimal

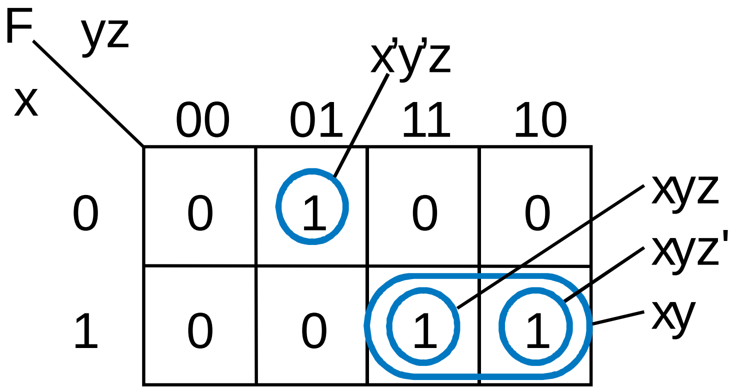


4 terms

Only 3 terms

Basic Concepts Underlying Automated Two-Level Logic Size Optimization

- Definitions
  - ***On-set***: All minterms that define when F=1
  - ***Off-set***: All minterms that define when F=0
  - ***Implicant***: Any product term (minterm or other) that when 1 causes F=1
    - On K-map, any legal (but not necessarily largest) circle
    - Cover: Implicant xy ***covers*** minterms xyz and xyz’
  - ***Expanding*** a term: removing a variable (like larger K-map circle)
    - xyz → xy is an expansion of xyz



x \ yz	00	01	11	10
0	0	1	0	0
1	0	0	1	1

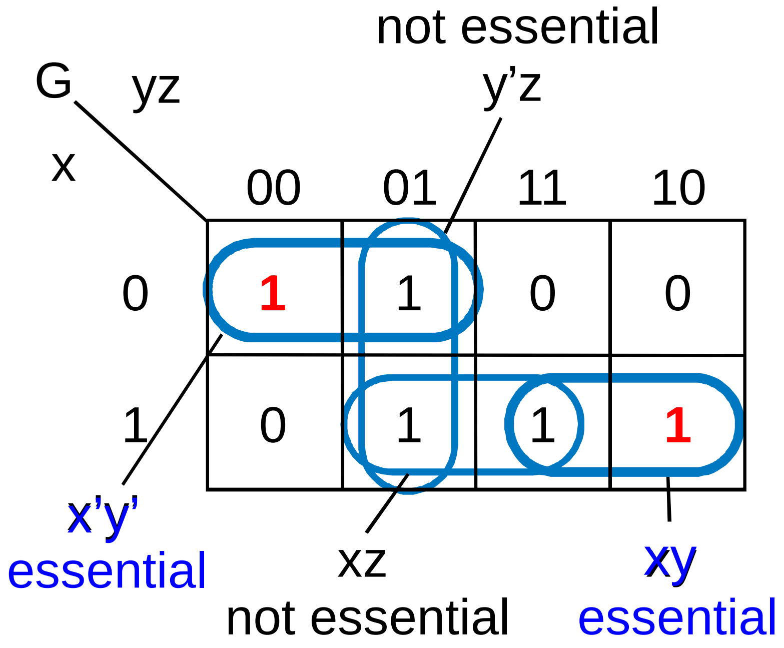
4 implicants of F

*Note: We use K-maps here just for intuitive illustration of concepts; automated tools do **not** use K-maps.*

- ***Prime implicant***: Maximally expanded implicant – any expansion would cover 1s not in on-set
  - x’y’z, and xy, above
  - But not xyz or xyz’ – they can be expanded

Basic Concepts Underlying Automated Two-Level Logic Size Optimization

- Definitions (cont)
  - ***Essential prime implicant***: The only prime implicant that covers a particular minterm in a function's on-set
    - Importance: We ***must*** include ***all*** essential PIs in a function's cover
    - In contrast, some, but not all, non-essential PIs will be included



G x \ yz	00	01	11	10
0	**1**	1	0	0
1	0	1	1	**1**

not essential
y'z

x'y'
essential

xz
not essential

xy
essential

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Automated Two-Level Logic Size Optimization Method

TABLE 6.1 Automatable tabular method for two-level logic size optimization.

	Step	Description
1	*Determine prime implicants*	Starting with minterm implicants, methodically compare all pairs (actually, all pairs whose numbers of uncomplemented literals differ by one) to find opportunities to combine terms to eliminate a variable, yielding new implicants with one less literal. Repeat for new implicants. Stop when no implicants can be combined. All implicants not covered by a new implicant are prime implicants.
2	*Add essential prime implicants to the function's cover*	Find every minterm covered by only one prime implicant, and denote that prime implicant as essential. Add essential prime implicants to the cover, and mark all minterms covered by those implicants as already covered.
3	*Cover remaining minterms with nonessential prime implicants*	Cover the remaining minterms using the minimal number of remaining prime implicants.

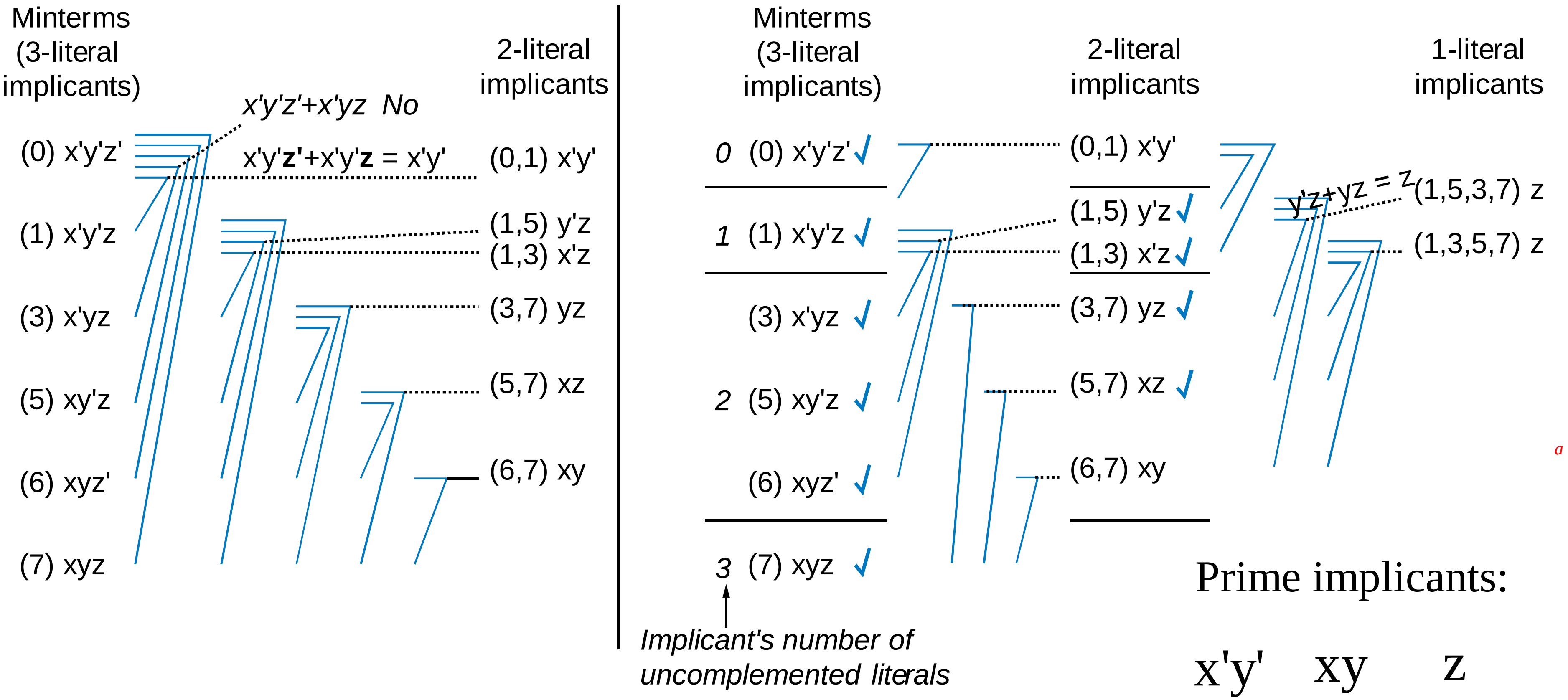
- Steps 1 and 2 are exact
- Step 3: Hard. Checking all possibilities: exact, but computationally expensive. Checking some but not all: heuristic.

Tabular Method Step 1: Determine Prime Implicants

Methodically Compare All Implicant Pairs, Try to Combine

- Example function: F = x'y'z' + x'y'z + x'yz + xy'z + xyz' + xyz

Actually, comparing ALL pairs isn't necessary—just pairs differing in uncomplemented literals by one.



Minterms (3-literal implicants)

2-literal implicants

x'y'z'+x'yz No

x'y'**z'**+x'y'**z** = x'y'

(0) x'y'z'
(1) x'y'z
(3) x'yz
(5) xy'z
(6) xyz'
(7) xyz

(0,1) x'y'
(1,5) y'z
(1,3) x'z
(3,7) yz
(5,7) xz
(6,7) xy

Minterms (3-literal implicants)

2-literal implicants

1-literal implicants

0 (0) x'y'z' ✓
1 (1) x'y'z ✓
(3) x'yz ✓
2 (5) xy'z ✓
(6) xyz' ✓
3 (7) xyz ✓

(0,1) x'y'
(1,5) y'z ✓
(1,3) x'z ✓
(3,7) yz ✓
(5,7) xz ✓
(6,7) xy

y'z+yz = z

(1,5,3,7) z
(1,3,5,7) z

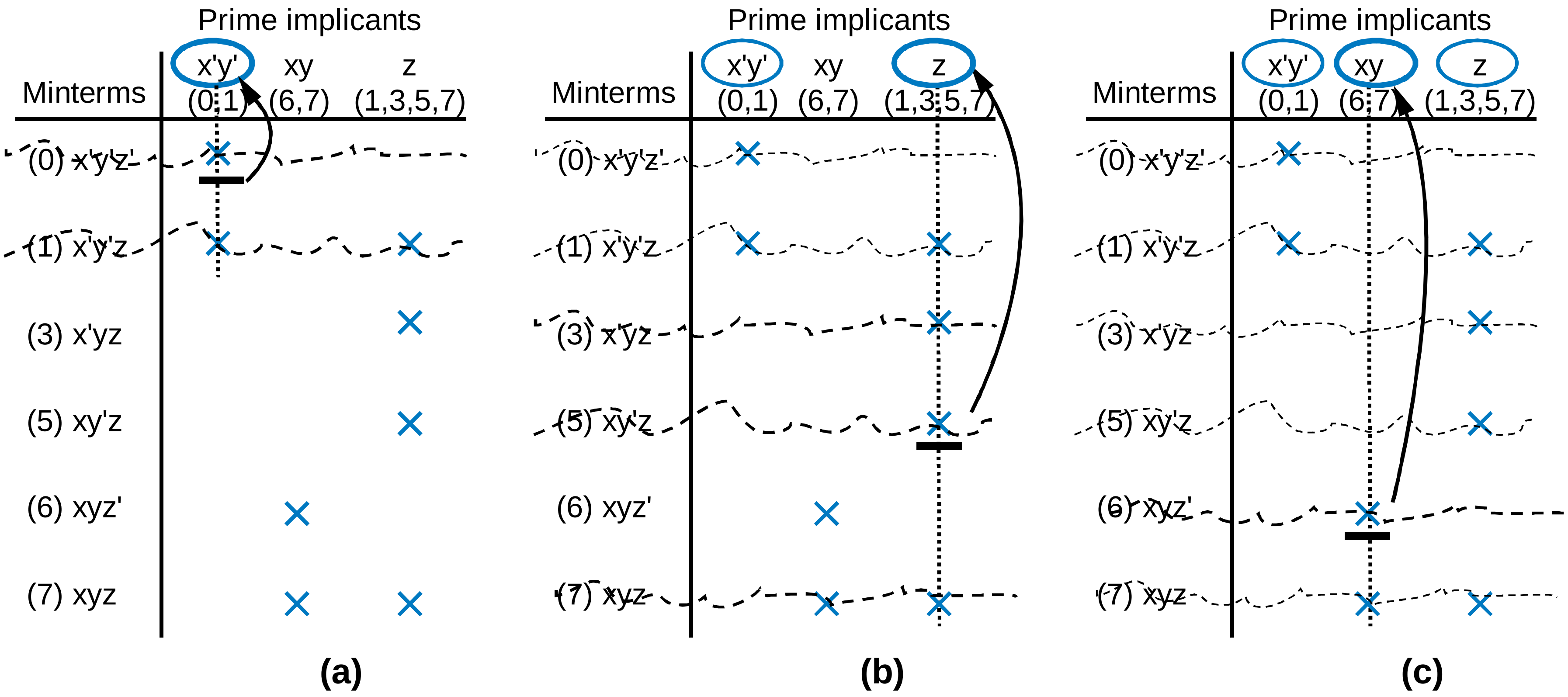
Implicant's number of uncomplemented literals

Prime implicants:

x'y' xy z

Tabular Method Step 2: Add Essential PIs to Cover

- Prime implicants (from Step 1): x'y', xy, z

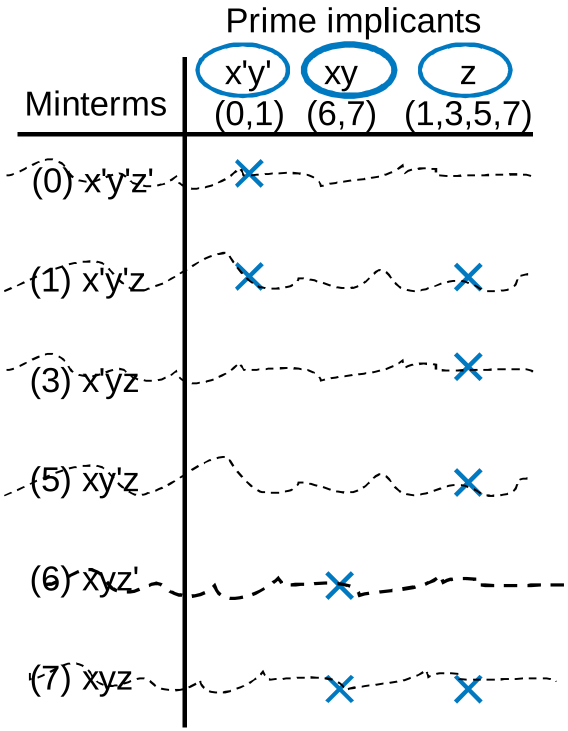


If only one **X** in row, then that PI is essential—it's the only PI that covers that row's minterm.

Tabular Method Step 3: Use Fewest Remaining PIs to Cover Remaining Minterms

- Essential PIs (from Step 2): x'y', xy, z
  - Covered all minterms, thus nothing to do in step 3
- Final minimized equation:

$$F = x'y' + xy + z$$



Prime implicants

Minterms	x'y' (0,1)	xy (6,7)	z (1,3,5,7)
(0) x'y'z'	✕		
(1) x'y'z	✕		✕
(3) x'yz			✕
(5) xy'z			✕
(6) xyz'		✕	
(7) xyz		✕	✕

(c)

Problem with Methods that Enumerate all Minterms or Compute all Prime Implicants

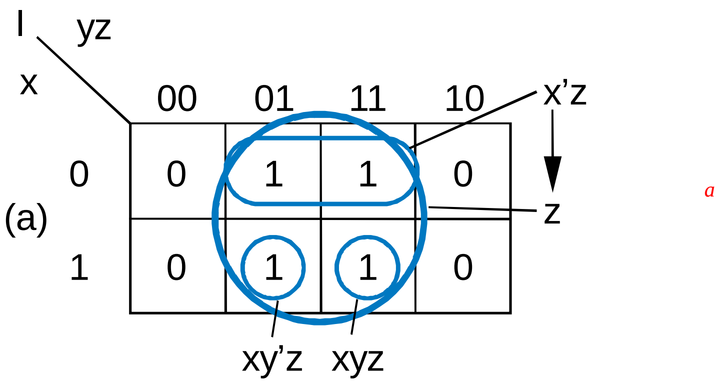
- Too many minterms for functions with many variables
  - Function with 32 variables:
    - 2^{32} = 4 billion possible minterms.
    - Too much compute time/memory
- Too many computations to generate all prime implicants
  - Comparing every minterm with every other minterm, for 32 variables, is (4 billion)^2 = 1 quadrillion computations
  - Functions with many variables could requires days, months, years, or more of computation – unreasonable

Solution to Computation Problem

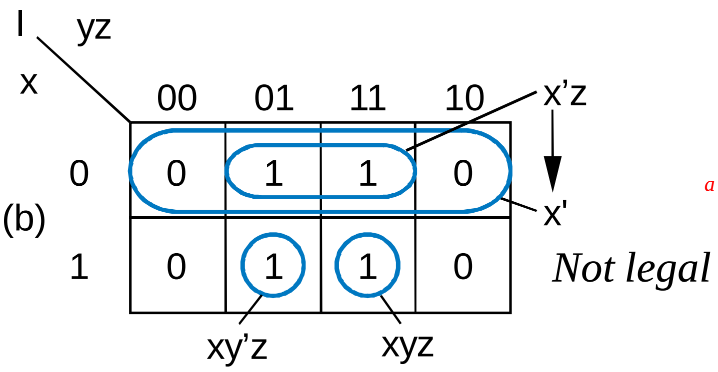
- Solution
  - Don’t generate all minterms or prime implicants
  - Instead, just take input equation, and try to “iteratively” improve it
  - Ex: F = abcdefgh + abcdefgh’+ jklmnop
    - Note: 15 variables, may have thousands of minterms
    - But can minimize just by combining first two terms:
      - F = abcdefg(h+h’) + jklmnop = abcdefg + jklmnop

Two-Level Optimization using Iterative Method

- Method: Randomly apply “expand” operations, see if helps
  - Expand: remove a variable from a term
    - Like expanding circle size on K-map
      - e.g., Expanding x’z to z legal, but expanding x’z to z’ not legal, in shown function
      - After expand, remove other terms covered by newly expanded term
  - Keep trying (iterate) until doesn’t help



(a)



(b)

Not legal

Ex:
F = abcdefgh + abcdefgh’+ jklmnop
F = abcdefg + abcdefgh’ + jklmnop
F = abcdefg + jklmnop

Illustrated above on K-map, but iterative method is intended for an automated solution (no K-map)

Covered by newly expanded term abcdefg

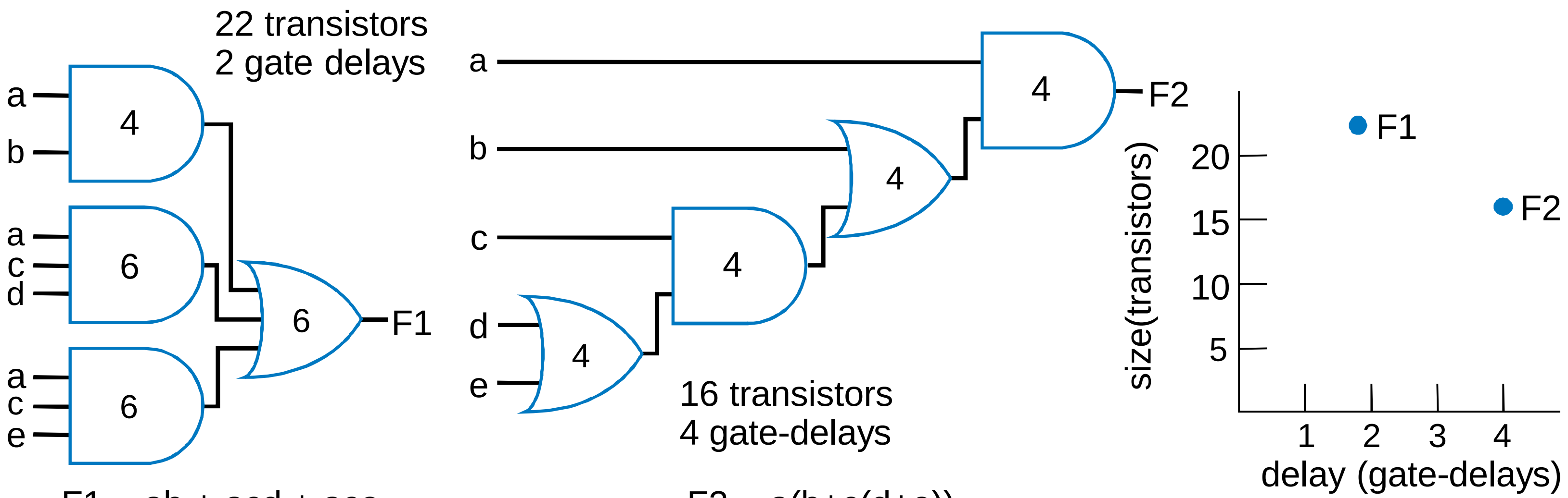
Ex: Iterative Hueristic for Two-Level Logic Size Optimization

- F = xyz + xyz' + x'y'z' + x'y'z *(minterms in on-set)*
- Random expand: F = xy~~z~~ + xyz' + x'y'z’ + x'y'z
  - Legal: Covers xyz' and xyz, both in on-set
  - Any implicant covered by xy? Yes, xyz'.
- F = xy + ~~xyz'~~ + x'y'z' + x'y'z
- Random expand: F = x~~y~~ + x'y'z' + x'y'z
  - Not legal (x covers xy'z', xy'z, xyz', xyz: two not in on-set)
- Random expand: F = xy + x'y'~~z'~~ + x'y'z
  - Legal
  - Implicant covered by x'y': x'y'z
- F = xy + x'y'z' + ~~x'y'z~~

a

Multi-Level Logic Optimization – Performance/Size Tradeoffs

- We don’t always need the speed of two-level logic
  - Multiple levels may yield fewer gates
  - Example
    - F1 = ab + acd + ace → F2 = ab + ac(d + e) = a(b + c(d + e))
    - General technique: Factor out literals – xy + xz = x(y+z)



F1 = ab + acd + ace
(a)

F2 = a(b+c(d+e))
(b)

(c)

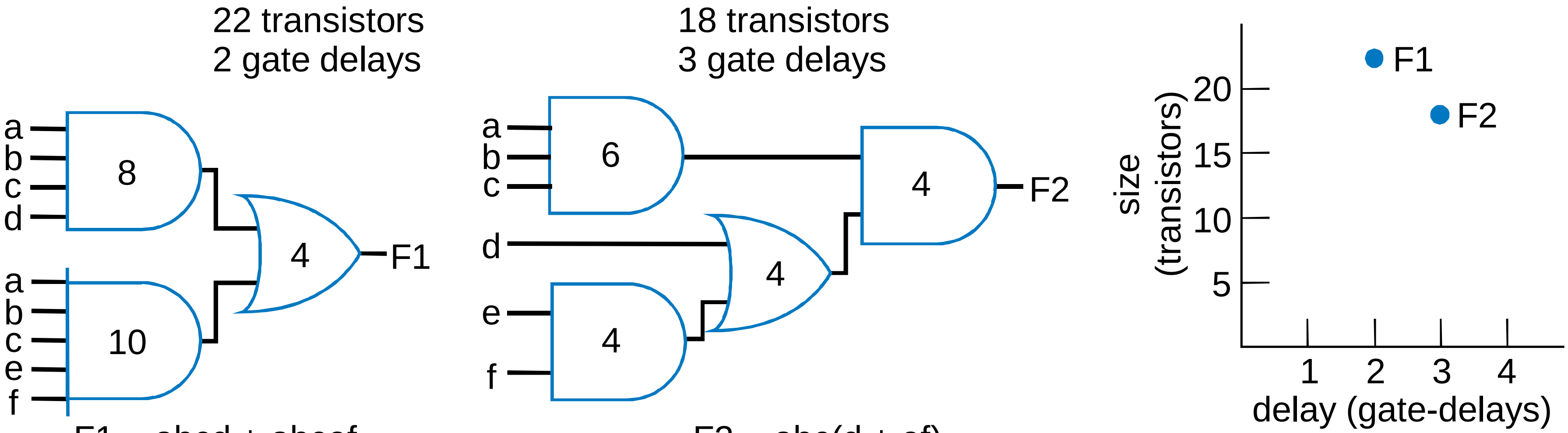
a

a

Multi-Level Example

- Q: Use multiple levels to reduce number of transistors for
  - – F1 = abcd + abcef
- A: abcd + abcef = abc(d + ef)
  - Has fewer gate inputs, thus fewer transistors

a



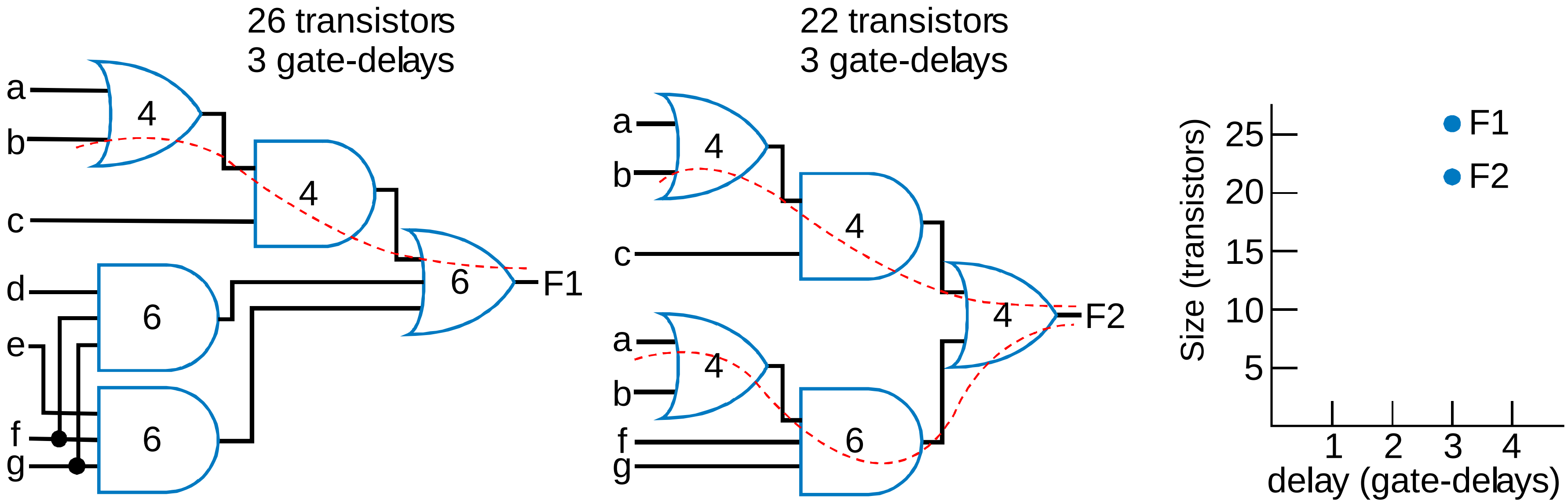
F1 = abcd + abcef
(a)

F2 = abc(d + ef)
(b)

(c)

Multi-Level Example: Non-Critical Path

- *Critical path*: longest delay path to output
- Optimization: reduce size of logic on non-critical paths by using multiple levels



26 transistors
3 gate-delays

22 transistors
3 gate-delays

F1 = (a+b)c + *dfg* + *efg*
(a)

F2 = (a+b)c + *(d+e)fg*
(b)

(c)

Automated Multi-Level Methods

- Main techniques use heuristic iterative methods
  - Define various operations
    - “Factoring”: abc + abd = ab(c+d)
    - Plus other transformations similar to two-level iterative improvement