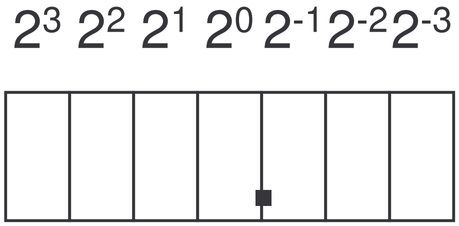
Floating Point Math, Fixed-Point Math and Other Optimizations

Floating Point (a brief look)

We need a way to represent

- numbers with fractions, e.g., 3.1415926535897932384626442
- very small numbers, e.g., .000000001
- very large numbers, e.g., 3.15576 × 10^9

Numbers with fractions:



Big and small numbers

Solution for decimal - scientific notation

- 2.5×10^{15} = 2,500,000,000,000,000

Can do the same for binary:

- 2MB = 2×2^{20} or $10_2 \times 10_2^{10100}$
- This is called a *floating-point number*
- In general, composed of sign, exponent, significand:
  $(-1)^{\text{sign}} \times \text{significand} \times 2^{\text{exponent}}$
- more bits for significand gives more accuracy
- more bits for exponent increases range

IEEE 754 floating point standard:

- single precision: 8 bit exponent, 23 bit significand
- double precision: 11 bit exponent, 52 bit significand

IEEE 754 floating-point standard

Leading “1” bit of significand is implicit

Exponent is “biased” to make sorting easier

- Biasing is a way of representing negative numbers
- All 0s is smallest exponent all 1s is largest
- bias of 127 for single precision and 1023 for double precision
- summary: $(-1)^{\text{sign}} \times (1+\text{significand}) \times 2^{\text{exponent} - \text{bias}}$

Example:

- decimal: -.75 = -3/4 = $-3/2^2$
- binary: -.11 = -1.1×2^{-1}
- floating point: exponent = 126 = 01111110
- IEEE single precision:
  `10111111010000000000000000000000`

Examples of Floating Point Numbers

Show the IEEE 754 binary representation for the number 20.0 in single and double precision:

$20 = 10100 \text{ x } 2^0$ or $1.0100 \text{ x } 2^4$

Single Precision:

The exponent is 127+4 = 131 = 128 + 3 = 10000011

The entire number is

0 1000 0011 0100 0000 0000 0000 0000 000

Double Precision:

The exponent is 1023+4 =1027=1024 + 3 =10000000011

The entire number is:

0 1000 0000 011 0100 0000 0000 0000 0000 0000   0000 0000 0000 0000 0000 0000 0000

Examples of Floating Point Numbers

Show the IEEE 754 binary representation for the number 20.5_{10} in single and double precision:

$20.0 = 10100 \times 2^0$, $0.5 = 0.1 \times 2^0$ together

1.01001×2^4

Single Precision:

The exponent is 127+4 = 131 = 128 + 3 = 10000011

The entire number is

0 1000 0011 0100 1000 0000 0000 0000 000

Double Precision:

The exponent is 1023+4 =1027=1024 + 3 =10000000011

The entire number is:

0 1000 0000 011 0100 1000 0000 0000 0000 0000 0000 0000 0000 0000 0000 0000 0000

Examples of Floating Point Numbers

Show the IEEE 754 binary representation for the number -0.1_{10} in single and double precision:

$0.1 = 0.0\underline{0011}$ x 2^0 or $1.1\underline{0011}$ x 2^{-4}

(the $\underline{0011}$ pattern is repeated)

Single Precision:

The exponent is 127-4 = 123 = 0111 1011

The entire number is

1 0111 1011 1001 1001 1001 1001 1001 100

Double Precision:

The exponent is 1023-4 = 1019 = 01111111011

The entire number is:

1 0111 1111 011 1001 1001 1001 1001 1001 1001 1001 1001 1001 1001 1001 1001 1001

Floating Point Complexities

Operations are somewhat more complicated (see text)

In addition to overflow we can have “underflow”

- Result of two adding two very small values becomes zero

Accuracy can be a big problem

- 1/(1/3) should = 3
- IEEE 754 keeps two extra bits, guard and round
- four rounding modes
- positive divided by zero yields “infinity”
- zero divide by zero yields “not a number” (NaN)

Implementing the standard can be tricky

Not using the standard can be even worse

Starting Points for Efficient Code

Write correct code first, optimize second.

Use a top-down approach.

Know your microprocessors’ architecture, compiler (features and also object model used), and programming language.

Leave assembly language for unported designs, interrupt service routines, and frequently used functions.

Floating Point Data Type Specifications

Use the smallest adequate data type, or else…

- Conversions without an FPU are very slow
- Extra space is used
- C standard allows compiler or preprocessor to convert automatically, slowing down code more

Single-precision (SP) vs. double-precision (DP)

- ANSI/IEEE 754-1985, *Standard for Binary Floating Point Arithmetic*
  - Single precision: 32 bits
    - 1 sign bit, 8 exponent bits, 23 fraction bits
  - Double precision
    - 1 sign bit, 11 exponent bits, 52 fraction bits
- Single-precision is likely all that is needed
- Use single-precision floating point specifier “f”

Floating-Point Specifier Example

No “f”

```
        float res = val / 10.0;
```

Assembler:

```
        move.l -4(a6),-(sp)
        jsr __ftod  //SP to DP
        clr.l -(sp)
        move.l #1076101120,-
(sp)
        jsr __ddiv  //DP divide
        jsr __dtof  //DP to SP
        move.l (s)+,-8(a6)
```

“f”

```
        float res = val / 10.0 f;
```

Assembler:

```
        move.l -4(a6),-(sp)
        move.l #1076101120,-
(sp)
        jsr __fdiv
        move.l (s)+,-8(a6)
```

Automatic Promotions

Standard math routines usually accept double-precision inputs and return double-precision outputs.

Again it is likely only single-precision is needed.

- Cast to single-precision if accuracy and overflow conditions are satisfied

Automatic Promotions Example

Automatic

```
            float res =
             (17.0f*sqrt(val)) / 10.0f;
// load val
// convert to DP
// _sqrt() on DP returns DP
// load DP of 17.0
// DP multiply
// load DP of 10.0
// DP divide
// convert DP to SP
// save result
```

two DP loads, DP multiply, DP divide, DP to SP conversion

Casting to avoid promotion

```
float res =
  (17.0f*(float)(sqrt(val)))/10.0f;
// load val
// convert to DP
// _sqrt() on DP returns DP
// convert result to SP
// load SP of 17.0
// SP multiply
// load SP of 10.0
// SP divide
// save result
```

two SP loads, SP multiply, SP divide

Rewriting and Rearranging Expressions

Divides take much longer to execute than multiplies.

Branches taken are usually faster than those not taken.

Repeated evaluation of same expression is a waste of time.

Examples

```
float res =  (17.0f*(float)(sqrt(val)))/10.0f;
```

is better written as:

```
float res = 1.7f*(float)(sqrt(val));
```

```
D = A / B / C;    as:  D = A / (B*C);

D = A / (B * C);       as:   bc = B * C;
    E = 1/(1+(B*C));           D =  A /  bc;
                                    E = 1/(1+bc);
```

Algebraic Simplifications and the Laws of Exponents

Original Expression	Optimized Expression
$a^2 – 3a + 2$ 2*, 1-, 1+	$(a – 1) * (a – 2)$ 1*, 2-
$(a - 1)*(a + 1)$ 1*, 1-, 1+	a^2-1 1*, 1-
$1/(1+a/b)$ 2/, 1+	$b/(b+a)$ 1/, 1+
$a^m * a^n$ 2 ^, 1*	a^{m+n} 1^, 1+
$(a^m)^n$ 2 ^	a^{m*n} 1^, 1*

Literal Definitions

#defines are prone to error

```
float c, r;
r = c/TWO_PI;
#define TWO_PI 2 * 3.14
```

- r = c/2*3.14 is wrong! Evaluates to r = (c/2) * 3.14

```
#define TWO_PI (2*3.14)
```

- Avoids a divide error
- However, 2*3.14 is loaded as DP leading to extra operations
  - convert c to DP
  - DP divide
  - convert quotient to SP

```
#define TWO_PI (float)(2.0*3.14)
```

- Avoids extra operations

The Standard Math Library Revisited

Double precision is likely expected by the standard math library.

Look for workarounds:

- abs()

```
float output=fabs(input);
```

could be written as:

```
if(input<0)
   output=-input;
else
   output=input;
```

Functions: Parameters and Variables

Consider a function which uses a global variable and calls other functions

- Compiler needs to save the variable before the call, in case it is used by the called function, reducing performance
- By loading the global into a local variable, it may be promoted into a register by the register allocator, resulting in better performance
  - Can only do this if the called functions don’t use the global

Taking the address of a local variable makes it more difficult or impossible to promote it into a register

More about Functions

Prototype your functions, or else the compiler may promote all arguments to ints or doubles!

Group function calls together, since they force global variables to be written back to memory

Make local functions static, keep in same module (source file)

- Allows more aggressive function inlining
- Only functions in same module can be inlined

Fixed Point Math – Why and How

Floating point is too slow and integers truncate the data

- Floating point subroutines: slower than native, overhead of passing arguments, calling subroutines… simple fixed point routines can be in-lined

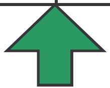
Basic Idea: put the *radix point* where covers the range of numbers you need to represent

I.F Terminology

- I = number of integer bits
- F= number of fraction bits

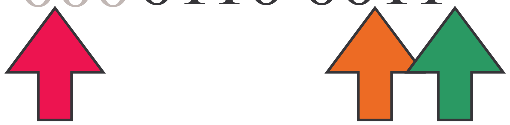
3.34375 in a fixed point binary representation

Bit	1	1	0	1	0	1	1
Weight	2^1	2^0	2^{-1}	2^{-2}	2^{-3}	2^{-4}	2^{-5}
Weight	2	1	½	¼	1/8	1/16	1/32



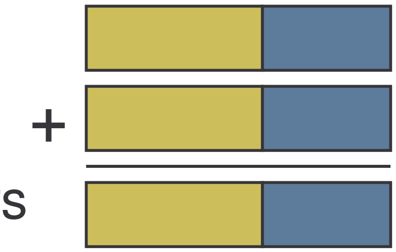
Radix Point

Bit Pattern	*Integer*	*6.2*	*1.10*
0000000 0000	0/1 = 0	0/4 = 0	0/1024 = 0
0000001 1100	28/1 = 28	28/4 = 7	28/1024 = 0.0273…
0000110 0011	99/1 = 99	99/4 = 24.75	99/1024 = 0.0966…



Radix Point Locations

Rules for Fixed Point Math

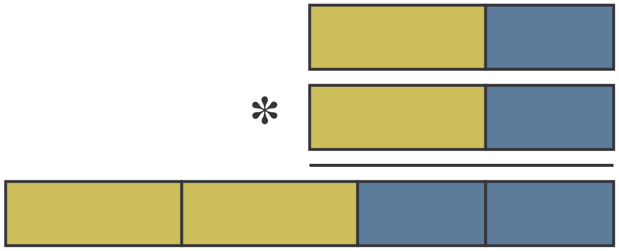


Addition, Subtraction

- Radix point stays where it started
- …so we can treat fixed point numbers like integers

Multiplication

- Radix point moves left by F digits
- … so we need to normalize result afterwards, shifting the result right by F digits

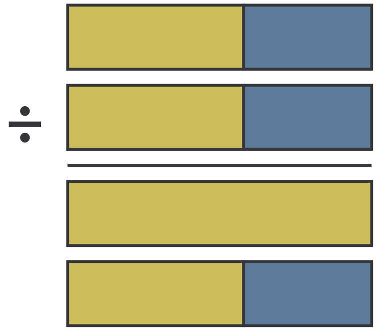


```
   10
* 1.25
  ----
  12.5
```

6.2 Format

```
            001010.00
           *000001.01
---------------------
00000000 1100.1000
```

Division

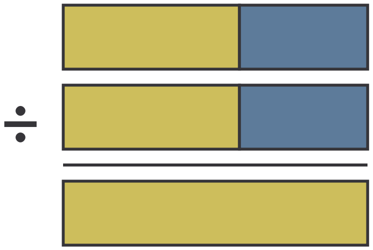


		3.1 Format	3.2 Format
Dividend	*7*	111.0	111.00
Divisor	÷ *2*	÷ 010.0	÷ 010.00
Quotient	*3*	0011	00011
Remainder	*1*	001.0	001.00

Division

- Quotient is *integer*, may want to convert back to fixed point by shifting
- Radix point doesn't move in remainder

Division, Part II



		3.1 Format	3.2 Format
Dividend	7	(7*2)1110.0	(7*4)11100.00
÷ *Divisor*	÷ 2	÷ 010.0	÷ 010.00
Quotient	3	0011.1	00011.10

Division

- To make quotient have same format as dividend and divisor, multiply dividend by 2^F (shift left by F bits)
- Quotient is in fixed point format now

Example Code for 12.4 Fixed Point Math

Representation

- `typedef unsigned int FX_12_4;`

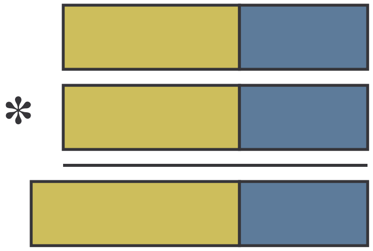
Converting to and from fixed point representation

- `#define FL_TO_FX(a) (unsigned int)(a*16.0)`
- `#define INT_TO_FX(a) (a<<4)`
- `#define FX_TO_FL(a) (float)(a/16.0)`
- `#define FX_TO_INT(a) (unsigned int)(a>>4)`

Math

- `#define FX_ADD(a,b) (a+b)`
- `#define FX_SUB(a,b) (a-b)`
- `#define FX_MUL(a,b) ((a*b)>>4)`
- `#define FX_DIV(a,b) ((a/b)<<4)`
- `#define FX_REM(a,b) ((a%b))`

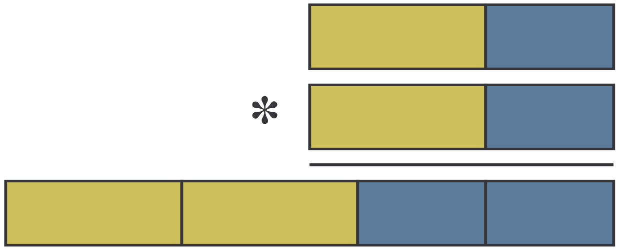
More Fixed Point Math Examples



```
   10
+ 1.5
 11.5
```

8.4 Format

```
         0000 1010.0000
        +0000 0001.1000
00000000 1100.1000
```



```
  9.0625
*    6.5
58.90625
```

4.4 Format

```
               1001.0001
              *0110.1000
0011 1010.1110 1000
```

Static Revisited

Static variable

- A local variable which retains its value between function invocations
- Visible only within its module, so compiler/linker can allocate space more wisely (recall limited pointer offsets)

Static function

- Visible only within its module, so compiler knows who is calling the functions,
- Compiler/linker can locate function to optimize calls (short call)
- Compiler can also inline the function to reduce run-time and often code size

Volatile and Const

Volatile variable

- Value can be changed outside normal program flow
  - ISR, variable is actually a hardware register
- Compiler reloads the variable from memory each time it is used

Const variable

- const does not mean constant, but rather read-only
- consts are implemented as real variables (taking space in RAM) or in ROM, requiring loading operations (often requiring pointer manipulation)
  - A #define value can be converted into an immediate operand, which is much faster
  - So avoid them

Const function parameters

- Allow compiler to optimize, as it knows a variable passed as a parameter hasn't been changed by the function

More

Const Volatile Variables

- Yes, it’s possible
  - Volatile: A memory location that can change unexpectedly
  - Const: it is only read by the program
- Example: hardware status register