The Alloy Model: Phase Transitions and Diagrams For a Random Energy Model with Mixtures

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Abstract

In this paper we introduce the alloy model, which is a variant of Derrida's random energy model (REM). The alloy model assumes that energy levels are independent and identically distributed (iid) random variables, whose distribution is a mixture of two distribution from the same location-scale family. These families are assumed to have either Weibull or Gumbel tails. Particular attention is paid to the case of normal distributions. For these we get more explicit results, which show that, for certain choices of the parameters, we can have two phase transitions, one first order and one second order.

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1 Introduction

The random energy model (REM) introduced by Derrida in [8] and [9] is one of the most famous mathematical models for a disordered system. Monographic reviews and many references are given in [5] and [15]. Some of the most important results about this model are related to phase transitions of the free energy in the thermodynamic limit. We recall that a phase transition is a point at which the function mapping the inverse temperature into

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the free energy is not infinitely differentiable. The order of a given phase transition is the smallest integer $n \geq 0$ such that the nth derivative of the function at this point does not exist. It is well-known that, for REM, the free energy in the thermodynamic limit has exactly one phase transition and it is of the second order. In this paper we introduce the alloy model, which generalizes REM by using mixtures. We find that, under certain conditions, the free energy may have two phase transitions, one first order and one second order. While, under other conditions, there may be no phase transition at all.

Recall that, in Derrida's random energy model, the partition function is given by

$$S_n(\beta) = \sum_{i=1}^{\lfloor e^n \rfloor} e^{\beta\sqrt{n}X_i},$$

where $\beta > 0$, $X_1, X_2, \ldots$ are independent and identically distributed (iid) standard normal $N(0,1)$ random variables, and $\lfloor \cdot \rfloor$ is the floor function. The parameter β is sometimes called the "inverse temperature" and is given by $\beta = 1/(k_B T)$, where T is the temperature and k_B is the Boltzmann constant. The random variables $\sqrt{n}X_i$ correspond to the (random) energy levels, and the value $\lfloor e^n \rfloor$ corresponds to the number of configurations. It is more common to let the number of configurations be 2^n, since this corresponds to the case where we have n particles and each can have either a positive or a negative spin. However, we make the slight modification of using $\lfloor e^n \rfloor$, because it has little effect on the underlying mathematics, but simplifies the formulas significantly.

The quantity

$$\mathcal{Z}_n(\beta) = \frac{\ln S_n(\beta)}{n}$$

is called the free energy. Some of the most important results about REM are related to phase transitions of the free energy in the thermodynamical limit, i.e. as $n \to \infty$. In particular, we have

$$\mathcal{Z}_n(\beta) \xrightarrow{p} \mathcal{Z}(\beta)$$

where $\xrightarrow{p}$ refers to convergence in probability and

$$\mathcal{Z}(\beta) = \begin{cases} 1 + \beta^2/2 & \text{if } \beta \leq \beta_c \\ \sqrt{2}\beta & \text{if } \beta \geq \beta_c \end{cases} \tag{1}$$

with $\beta_c = \sqrt{2}$. It is important to note that $\mathcal{Z}$ is a continuous and differentiable function of β. However, it does not have a second derivative at $\beta = \beta_c$. Thus $\mathcal{Z}$ has a second order phase transition at $\beta = \beta_c$. This result was first suggested by [9] and then rigorously proved in [10] and [17]. In addition, limit theorems for the partition function were studied in [6]. Further, limit theorems for the partition function, when the random variables are no longer assumed to be Gaussian, but have Weibull or Gumbel tails, were studied in [1], [2], and [4].

In this paper we introduce the alloy model, which is a generalization of REM based on mixtures. Specifically, let $X_1, X_2, \ldots \overset{\text{iid}}{\sim} N(0,1)$, fix $\sigma > 0$, $\eta \in \mathbb{R}$, $p \in (0,1)$, and let $Y_{1,n}$, $Y_{2,n}$,. . . ,$Y_{\lfloor e^n \rfloor,n}$ be iid random variables such that for each $n = 1, 2, \ldots$

$$Y_{i,n} = \begin{cases} \sqrt{n} X_i & \text{with probability } p \\ \sqrt{n}\sigma X_i + n\eta & \text{with probability } (1-p) \end{cases}.$$

In this case, the partition function and the free energy are given, respectively, by

$$\tilde{S}_n(\beta) = \sum_{i=1}^{\lfloor e^n \rfloor} e^{\beta Y_{i,n}} \text{ and } \tilde{\mathcal{Z}}_n(\beta) = \frac{\ln \tilde{S}_n(\beta)}{n}.$$

We study limit theorems of the free energy and find that, for certain choices of σ and η, we may have two phase transitions, one first order and one second order. Further, we consider not only the Gaussian case, but allow for a variety of distributions with Weibull or Gumbel tails. Interestingly, we find that there is no phase transition in the case of Gumbel tails. We leave the study of limit theorems of the partition function for future work, but note that, in the Gaussian case, such limit theorems were considered in [16].

The rest of the paper is organized as follows. In Section 2, we review basic facts about the class of regularly varying functions, which are important for discussing our models. In Section 3, we introduce the alloy model for distributions with Weibull tails, and study the behavior of the free energy. In Section 4, we extend these results to models with Gumbel tails. In Section 5, we give detailed results for the case of Gaussian distributions. Proofs are postponed to the Appendix.

2 Regularly Varying Functions

In this section we review basic facts about regularly varying functions. Recall that, for $\rho \in \mathbb{R}$, a measurable function $f : (0,\infty) \mapsto [0,\infty)$ is said to be regularly varying with index ρ if

$$\lim_{s\to\infty} \frac{f(xs)}{f(s)} = x^\rho \text{ for every } x > 0.$$

In this case we write $f \in R_\rho$. Let $f^{\leftarrow}(x) = \inf\{y > 0 : f(y) > x\}$ be the generalized inverse of f. By Theorem 1.5.12 in [3], if $\rho > 0$ then $f^{\leftarrow} \in R_{1/\rho}$ and

$$f(f^{\leftarrow}(x)) \sim f^{\leftarrow}(f(x)) \sim x \text{ as } x \to \infty. \tag{2}$$

It is well-known that every $f \in R_\rho$ can be represented as

$$f(x) = c(x) \exp\left\{ \int_a^x \frac{\rho + \epsilon(u)}{u} \mathrm{d}u \right\} \text{ for } x \geq a,$$

where $a > 0$, and c and ϵ are measurable functions satisfying $c(x) \to c \in (0, \infty)$ and $\epsilon(x) \to 0$ as $x \to \infty$. This is the so-called Karamata representation, see [3]. A function $f \in R_\rho$ is said to be normalized regularly varying if we can take $c(x) = c$ for all $x > 0$. Thus, its Karamata representation is of the form

$$f(x) = c \exp\left\{ \int_a^x \frac{\rho + \epsilon(u)}{u} \mathrm{d}u \right\} \text{ for } x \geq a,$$

where $c \in (0, \infty)$. In this case we write $f \in NR_\rho$. Clearly such functions are continuous. Further, if $\rho > 0$ then, for large enough u, $\rho + \epsilon(u) > 0$, which means that f is eventually strictly increasing. Thus, if $f \in NR_\rho$ and $\rho > 0$, then f is invertible for large enough x, and we write f^{-1} instead of $f^{\leftarrow}$. This means that there is an $X > 0$ such that if $x > X$ then

$$f(f^{-1}(x)) = f^{-1}(f(x)) = x. \tag{3}$$

This is a much stronger condition than (2). The following result characterizes normalized regularly varying functions, see [3].

Lemma 1. *A function $f \in R_\rho$ is normalized regularly varying if and only if it is differentiable almost everywhere and*

$$\lim_{x \to \infty} \frac{x f'(x)}{f(x)} = \rho.$$

3 Distributions with Weibull Tails

Let F be a cumulative distribution function (cdf) and let $X \sim F$. Throughout, we assume that

$$\mathrm{E}\left[e^{tX}\right] < \infty \text{ for all } t > 0. \tag{4}$$

Let

$$H(t) = \ln \mathrm{E}\left[e^{tX}\right] = \ln \int_{\mathbb{R}} e^{tx} \mathrm{d}F(x), \quad \text{for } t > 0$$

and

$$h(x) = -\ln P(X > x) = -\ln\left(1 - F(x)\right), \quad \text{for } x > 0.$$

In this section we consider distributions with some of the heaviest tails, that satisfy (4). These are distributions with Weibull tails. Later, in Section 4, we consider distributions with Gumbel tails, which are lighter.

3.1 Definition

We say that a distribution has Weibull tails if it satisfies:

Assumption W. Fix $\rho' > 1$, $a > 0$, and assume that $h \in NR_{\rho'}$ with

$$h(x) \sim ax^{\rho'} \text{ as } x \to \infty.$$

Remark 1. *Lemma 3.3 in [2] implies that, when Assumption W holds,*

$$H(t) \sim \gamma t^{\rho} \text{ as } t \to \infty, \tag{5}$$

where $\gamma > 0$ is a constant depending only on a and ρ' and

$$\rho = \frac{\rho'}{\rho' - 1} \in (1, \infty).$$

Most of our results will be given in terms of the parameters γ and ρ. Note that we can get ρ' back from ρ by taking $\rho' = \frac{\rho}{\rho-1}$.

Representative examples of distributions that satisfy Assumption W are Weibull distributions, which have cdfs of the form

$$F(x) = \begin{cases} 1 - e^{-ax^{\rho'}} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}.$$

Another important example is the standard normal distribution. The fact that it satisfies Assumption W is verified by the following.

Lemma 2. *For the standard normal distribution, $N(0,1)$, $h \in NR_2$ and*

$$h(x) \sim .5x^2 \text{ as } x \to \infty.$$

Further, in this case, $H(t) = .5t^2$.

Proof. The proof is given in Appendix A. □

3.2 Randomized REM

Let F be a cdf, which satisfies Assumption W, and let $X_1, X_2, \ldots \stackrel{\text{iid}}{\sim} F$. Let $R : [0, \infty) \mapsto \mathbb{N}$ be a function such that there is a constant $\lambda > 0$ with

$$\lim_{t \to \infty} R(t) e^{-\lambda \gamma t} = 1. \tag{6}$$

For $\beta > 0$, set

$$U_t(\beta) = \sum_{i=1}^{R(t)} e^{t^{1/\rho} \beta X_i}.$$

This is the partition function and $R(t)$ is the number of configurations. Limit theorems for U_t are given in [1] and [2]. We consider a more general situations, where the number of terms in the sum, i.e. the number of configurations, is also random. We call this situation randomized REM. Understanding randomized REM is important for studying the alloy model.

Let $\{M_t : t \geq 0\}$ be a collection of $\mathbb{N}$-valued random variables independent of the sequence $\{X_n\}$. We allow the distributions of the M_t's to be point-masses, thus deterministic functions are a special case of this situation. For $\beta > 0$ set

$$S_t(\beta) = \sum_{i=1}^{M_t} e^{t^{1/\rho}\beta X_i},$$

and assume that

$$\frac{M_t}{R(t)} \stackrel{p}{\rightarrow} c \in (0,\infty) \text{ as } t \to \infty,$$

where R satisfies (6). Limit theorems for $S_t(\beta)$ are given in Appendix A. Here, we concern ourselves with the free energy, which is given by

$$\mathcal{Z}_t(\beta) = \frac{\ln S_t(\beta)}{t}.$$

Theorem 1. *Let* $\beta_* = (\frac{\lambda}{\rho-1})^{1/\rho}$. *If* $\lambda \in (0,\infty)$ *then*

$$\frac{\ln S_t(\beta)}{t} \stackrel{p}{\rightarrow} G(\beta),$$

where

$$G(\beta) = \begin{cases} \gamma\lambda + \gamma\beta^\rho & \text{if } \beta \leq \beta_* \\ \gamma\rho\beta\beta_*^{\rho-1} & \text{if } \beta \geq \beta_* \end{cases}.$$

Note that the value of c does not affect this result.

Proof. This follows immediately from Proposition 5, which is given in Appendix A. □

Remark 2. *By Lemma 2, Theorem 1 applies to the standard normal distribution. In particular, Derrida's REM, as described in Section 1, corresponds to the case, where* $\rho = 2$, $\gamma = .5$, *and* $M_t = R(t) = \lfloor e^t \rfloor$. *In this case* $\lambda = 2$ *and* $G(\beta)$ *reduces to* (1).

Remark 3. *The proof of Theorem 1 follows from Proposition 5, which is itself proved using the limit theorems for the partition function given in [1] and [2]. A different approach to proving such results is given in the PhD thesis [13], where a related result is proved using techniques from large deviations. While the assumptions in the thesis are quite different, [13] shows that they hold for certain distributions with Weibull tails. We thank the anonymous referee for bringing our attention to [13].*

We can, immediately, extend Theorem 1 to location-scale families as follows.

Corollary 1. *Fix $\eta \in \mathbb{R}$, $\sigma > 0$, and define*

$$S_t^{(\eta,\sigma)}(\beta) = \sum_{i=1}^{M_t} e^{\beta t^{1/\rho}\left(\sigma X_i + \eta t^{1-1/\rho}\right)} = S_t(\beta\sigma)e^{\beta\eta t}.$$

In this case

$$\frac{\ln S_t^{(\eta,\sigma)}(\beta)}{t} = \beta\eta + \frac{\ln S_t(\beta\sigma)}{t} \xrightarrow{p} \beta\eta + G_\rho(\beta\sigma).$$

3.3 The Alloy Model

Let F be a cdf, which satisfies Assumption W. Fix $p \in (0,1)$, $\eta \in \mathbb{R}$, and $\sigma > 0$. For $t > 0$, let $F_p^{(t)}$ be the cdf given by

$$F_p^{(t)}(x) = pF(x) + (1-p)F\left(\frac{x - \eta t^{1-1/\rho}}{\sigma}\right).$$

Thus F_p is the mixture of two distributions from the same location-scale family. We call this the alloy model for distributions with Weibull tails.

Let $R : [0,\infty) \mapsto \mathbb{N}$ and assume that there is a constant $\lambda > 0$ satisfying (6). Fix $t > 0$, let $X_1^{(t)}, X_2^{(t)}, \ldots \stackrel{\text{iid}}{\sim} F_p^{(t)}$, and define

$$\tilde{S}_t(\beta) = \sum_{i=1}^{R(t)} e^{t^{1/\rho}\beta X_i^{(t)}}$$

and

$$\tilde{\mathcal{Z}}_t(\beta) = \frac{\ln \tilde{S}_t(\beta)}{t}.$$

These are, respectively, the partition function and the free energy for the alloy model.

To characterize the limits of $\tilde{\mathcal{Z}}_t(\beta)$, it is convenient to put everything into a slightly different form. Let $Y_1, Y_2, \ldots \stackrel{\text{iid}}{\sim} F$ and $Y_1', Y_2', \ldots \stackrel{\text{iid}}{\sim} F$ be independent sequences. Independent of these, let $M_t \sim B(R(t), p)$, where $B(n,p)$ represents a binomial distribution with parameters n and p. Let

$$\begin{aligned} S_t^{(1)}(\beta) &= \sum_{i=1}^{M_t} e^{t^{1/\rho}\beta Y_i}, \\ S_t^{(2)}(\beta) &= \sum_{i=1}^{R(t)-M_t} e^{t^{1/\rho}\beta\sigma Y_i' + \beta\eta t}, \end{aligned}$$

and set

$$\mathcal{Z}_t^* = \frac{\ln\left(S_t^{(1)}(\beta) + S_t^{(2)}(\beta)\right)}{t}.$$

Note that

$$\tilde{S}_t(\beta) \stackrel{d}{=} S_t^{(1)}(\beta) + S_t^{(2)}(\beta)$$

and

$$\tilde{\mathcal{Z}}_t(\beta) \stackrel{d}{=} \tilde{\mathcal{Z}}_t^*(\beta). \tag{7}$$

For $i = 1, 2$ set

$$\mathcal{Z}_t^{(i)}(\beta) = \frac{\ln S_t^{(i)}(\beta)}{t}.$$

By the law of large numbers, we have

$$\frac{M_t}{R(t)} \xrightarrow{p} p \text{ and } \frac{R(t) - M_t}{R(t)} \xrightarrow{p} (1-p).$$

Thus, by Theorem 1 and Corollary 1, it follows that, if

$$\mathcal{Z}^{(1)}(\beta) = G(\beta) \text{ and } \mathcal{Z}^{(2)}(\beta) = \beta\eta + G(\beta\sigma),$$

then for $i = 1, 2$ we have

$$\mathcal{Z}_t^{(i)}(\beta) \xrightarrow{p} \mathcal{Z}^{(i)}(\beta) \text{ as } t \to \infty.$$

Now set

$$\mathcal{Z}_t^{\vee}(\beta) = \max\left\{\mathcal{Z}_t^{(1)}(\beta), \mathcal{Z}_t^{(2)}(\beta)\right\}$$

and note that, by the continuous mapping theorem and the fact that the maximum of two functions is continuous, it follows that

$$\mathcal{Z}_t^{\vee}(\beta) \xrightarrow{p} \max\left\{\mathcal{Z}^{(1)}(\beta), \mathcal{Z}^{(2)}(\beta)\right\}.$$

Next, note that, by positivity

$$\max\{S_t^{(1)}(\beta), S_t^{(2)}(\beta)\} \leq S_t^{(1)}(\beta) + S_t^{(2)}(\beta) \leq 2\max\{S_t^{(1)}(\beta), S_t^{(2)}(\beta)\}$$

and hence

$$\frac{\ln\left(\max\{S_t^{(1)}(\beta), S_t^{(2)}(\beta)\}\right)}{t} \leq \tilde{\mathcal{Z}}_t^*(\beta) \leq \frac{\ln\left(\max\{S_t^{(1)}(\beta), S_t^{(2)}(\beta)\}\right) + \ln 2}{t},$$

or, equivalently,

$$\mathcal{Z}_t^{\vee}(\beta) \leq \tilde{\mathcal{Z}}_t^*(\beta) \leq \mathcal{Z}_t^{\vee}(\beta) + \frac{\ln 2}{t}.$$

From here, by the squeeze theorem for convergence in probability, it follows that

$$\tilde{\mathcal{Z}}_t^*(\beta) \xrightarrow{p} \max\left\{\mathcal{Z}^{(1)}(\beta), \mathcal{Z}^{(2)}(\beta)\right\}.$$

Combining this, with (7) and the facts that

$$\mathcal{Z}^{(1)}(\beta) = \begin{cases} \gamma\lambda + \gamma\beta^\rho & \text{if } \beta \le \beta_c^{(1)} \\ \gamma\rho\beta\left(\beta_c^{(1)}\right)^{\rho-1} & \text{if } \beta \ge \beta_c^{(1)} \end{cases}$$

and

$$\mathcal{Z}^{(2)}(\beta) = \begin{cases} \beta\eta + \gamma\lambda + \gamma\beta^\rho\sigma^\rho & \text{if } \beta \le \beta_c^{(2)} \\ \beta\eta + \gamma\rho\beta\sigma\left(\beta_c^{(1)}\right)^{\rho-1} & \text{if } \beta \ge \beta_c^{(2)} \end{cases},$$

where

$$\beta_c^{(1)} = \left(\frac{\lambda}{\rho-1}\right)^{1/\rho} \text{ and } \beta_c^{(2)} = \sigma^{-1}\left(\frac{\lambda}{\rho-1}\right)^{1/\rho} = \sigma^{-1}\beta_c^{(1)} \tag{8}$$

gives the following result.

Proposition 1. *For every* $\beta > 0$

$$\tilde{\mathcal{Z}}_t(\beta) \xrightarrow{p} \tilde{\mathcal{Z}}(\beta),$$

where $\tilde{\mathcal{Z}}(\beta)$ *is as follows. If* $\sigma > 1$ *then*

$$\tilde{\mathcal{Z}}(\beta) = \begin{cases} \gamma\lambda + \max\left\{\gamma\beta^\rho, \beta\eta + \gamma\beta^\rho\sigma^\rho\right\} & \text{if } \beta \le \beta_c^{(2)} \\ \max\left\{\gamma\lambda + \gamma\beta^\rho, \beta\eta + \gamma\rho\beta\sigma\left(\beta_c^{(1)}\right)^{\rho-1}\right\} & \text{if } \beta_c^{(2)} \le \beta \le \beta_c^{(1)} \\ \beta\max\left\{\gamma\rho\left(\beta_c^{(1)}\right)^{\rho-1}, \eta + \gamma\rho\sigma\left(\beta_c^{(1)}\right)^{\rho-1}\right\} & \text{if } \beta \ge \beta_c^{(1)} \end{cases}.$$

and if $\sigma \in (0,1)$ *then*

$$\tilde{\mathcal{Z}}(\beta) = \begin{cases} \gamma\lambda + \max\left\{\gamma\beta^\rho, \beta\eta + \gamma\beta^\rho\sigma^\rho\right\} & \text{if } \beta \le \beta_c^{(1)} \\ \max\left\{\gamma\beta\rho\left(\beta_c^{(1)}\right)^{\rho-1}, \beta\eta + \gamma\lambda + \gamma\beta^\rho\sigma^\rho\right\} & \text{if } \beta_c^{(1)} \le \beta \le \beta_c^{(2)} \\ \beta\max\left\{\gamma\rho\left(\beta_c^{(1)}\right)^{\rho-1}, \eta + \gamma\sigma\rho\left(\beta_c^{(1)}\right)^{\rho-1}\right\} & \text{if } \beta \ge \beta_c^{(2)} \end{cases}.$$

Here, $\beta_c^{(1)}$ *and* $\beta_c^{(2)}$ *are as in* (8).

Note that the parameter p does not affect this result. While, the formula for $\tilde{\mathcal{Z}}(\beta)$, given by Proposition 1, is straightforward to evaluate numerically, one cannot, in general, solve the maximization analytically. However, we will see that there is an important situation, where an explicit solution is possible, see Section 5 below.

4 Distributions with Gumbel Tails

In this section we consider a model with tails that are lighter than those allowed by the Weibull model. Specifically, these are models with Gumbel tails. Limit theorems for the partition function in this case were studied in [4]. We say that a distribution has Gumbel tails if it satisfies:

Assumption G. Let $g(x) = h(\ln x)$ and fix $\gamma, a > 0$. Assume that $g \in NR_{1/\gamma}$ and that

$$g(x) \sim ax^{1/\gamma}, \text{ as } x \to \infty.$$

Remark 4. *This implies that*

$$h(x) \sim ae^{x/\gamma}, \text{ as } x \to \infty.$$

By the discussion in Section 2, $g(x)$ is invertible for large enough x, and, by Lemma 3 in Appendix B,

$$g^{-1}(x) \sim (x/a)^{\gamma} \text{ as } x \to \infty. \tag{9}$$

Remark 5. *Theorem 4 in [14] (see also Lemma 3.3 in [4]) relates the behavior of g^{-1} to that of H. Specifically, it shows that*

$$\frac{H(t)}{t} - \ln(g^{-1}(t)) \to \gamma(\ln\gamma - 1) \text{ as } t \to \infty. \tag{10}$$

Typical examples of distributions that satisfy Assumption G are the reverse Gumbel distribution, which has a cdf given by

$$F(x) = 1 - e^{-ae^{x/\gamma}},$$

and the Gompertz distribution, which has a cdf given by

$$F(x) = \begin{cases} 1 - e^{-ae^{x/\gamma}+a} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}.$$

Let $u : (0, \infty) \mapsto (0, \infty)$ be the inverse of the function that maps $t \mapsto t\ln t$. This means that

$$u(t) = e^{W(t)},$$

where W is the Lambert W function. Recall that W is the unique solution to the functional equation $W(t)e^{W(t)} = t$, $t > 0$. It follows that $e^{W(t)} \ln e^{W(t)} = t$, which gives the form of u. For additional properties and applications of Lambert's W function see the classic survey [7]. By change of variables we have

$$\lim_{t\to\infty} \frac{\frac{t}{\ln t}}{u(t)} = \lim_{t\to\infty} \frac{\frac{t\ln t}{\ln t+\ln\ln t}}{t} = \lim_{t\to\infty} \frac{\ln t}{\ln t + \ln\ln t} = 1.$$

Thus $u(t) \sim t/\ln t$ as $t \to \infty$, and hence

$$\lim_{t\to\infty} u(t) = \infty.$$

Let $X_1, X_2, \ldots \overset{\text{iid}}{\sim} F$, where F is a cdf satisfying Assumption G. Let $\{M_t : t \geq 0\}$ be a collection of $\mathbb{N}$-valued random variables independent of the sequence $\{X_n\}$. For $\beta > 0$ set

$$S_t(\beta) = \sum_{i=1}^{M_t} e^{u(t)\beta X_i}$$

and

$$\mathcal{Z}_t(\beta) = \frac{\ln S_t(\beta)}{t}.$$

Assume that there exists a deterministic function $R : [0,\infty) \mapsto \mathbb{N}$ such that, for some $\lambda > 0$

$$\lim_{t\to\infty} R(t\ln t)e^{-\lambda\gamma t} = 1 \tag{11}$$

and for some $c \in (0,\infty)$

$$\frac{M_t}{R(t)} \xrightarrow{p} c.$$

In this case, we get the following result.

Theorem 2. *If $\lambda \in (0,\infty)$ then*

$$\frac{\ln S_t(\beta)}{t} \xrightarrow{p} \gamma\beta.$$

Proof. The proof is given in Appendix B. □

We can, immediately, extend this result to location-scale families as follows.

Corollary 2. *Fix $\eta \in \mathbb{R}$, $\sigma > 0$, and define*

$$S_t^{(\eta,\sigma)}(\beta) = \sum_{i=1}^{M_t} e^{\beta u(t)(\sigma X_i + \eta t/u(t))} = S_t(\beta\sigma)e^{\beta\eta t}.$$

In this case

$$\frac{\ln S_t^{(\eta,\sigma)}(\beta)}{t} = \beta\eta + \frac{\ln S_t(\beta\sigma)}{t} \xrightarrow{p} (\gamma\sigma + \eta)\,\beta.$$

We now turn to the alloy model for this situation. Fix $p \in (0,1)$, $\eta \in \mathbb{R}$, and $\sigma > 0$. For $t > 0$ let $F_p^{(t)}$ be the cdf given by

$$F_p^{(t)}(x) = pF(x) + (1-p)F\left(\frac{x - \eta t/u(t)}{\sigma}\right).$$

Let $X_1^{(t)}, X_2^{(t)}, \ldots \overset{\text{iid}}{\sim} F_p^{(t)}$ and set

$$\tilde{S}_t(\beta) = \sum_{i=1}^{R(t)} e^{u(t)\beta X_i^{(t)}}$$

and

$$\tilde{\mathcal{Z}}_t(\beta) = \frac{\ln \tilde{S}_t}{t}.$$

By arguments similar to those in Section 3.3, we have

$$\tilde{\mathcal{Z}}_t(\beta) \xrightarrow{p} \tilde{\mathcal{Z}}(\beta) = \max\left\{\mathcal{Z}^{(1)}(\beta), \mathcal{Z}^{(2)}(\beta)\right\},$$

where

$$\mathcal{Z}^{(1)}(\beta) = \gamma\beta \text{ and } \mathcal{Z}^{(2)}(\beta) = (\gamma\sigma + \eta)\beta.$$

It follows that

$$\tilde{\mathcal{Z}}(\beta) = \max\{\gamma, \gamma\sigma + \eta\}\beta.$$

Note that, in this case, there is no phase transition.

5 Gaussian Alloy Model

In this section, we formally introduce the Gaussian alloy model. By Lemma 2 this is a model with Weibull tails, where $\gamma = .5$ and $\rho = 2$. Thus, the results of Section 3 hold. However, in this case, we can get an explicit solution for $\tilde{\mathcal{Z}}(\beta)$.

Fix $\sigma > 0$, $\eta \in \mathbb{R}$, and $p \in (0,1)$. Take $R(t) = \lfloor e^t \rfloor$ and let $X_1, X_2, \ldots \overset{\text{iid}}{\sim} N(0,1)$. For $t > 0$, define the random variables $Y_1^{(t)}$, $Y_2^{(t)}, \ldots, Y_{R(t)}^{(t)}$ to be given by

$$Y_i^{(t)} = \begin{cases} \sqrt{t}X_i & \text{with probability } p \\ \sqrt{t}\sigma X_i + \eta t & \text{with probability } (1-p) \end{cases},$$

where the choice is made independently for each i. This means that, for each t, the random variables $Y_1^{(t)}$, $Y_2^{(t)}, \ldots, Y_{R(t)}^{(t)}$ are iid with cdf

$$\Phi^{(t)}(x) = p\Phi\left(\frac{x}{\sqrt{t}}\right) + (1-p)\Phi\left(\frac{x - t\eta}{\sigma\sqrt{t}}\right),$$

where $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} \mathrm{d}t$ is the cdf of a $N(0,1)$ distribution. In this case the partition function is given by

$$S_t(\beta) = \sum_{i=1}^{\lfloor e^t \rfloor} e^{\beta Y_i^{(t)}}.$$

Note that, if we restrict t to the integers, then this is exactly the model introduced in Section 1. Proposition 1 implies that the free energy satisfies

$$\mathcal{Z}_t = \frac{\ln S_t(\beta)}{t} \xrightarrow{p} F_{\sigma,\eta}(\beta),$$

where $F_{\sigma,\eta}(\beta) = \tilde{\mathcal{Z}}(\beta)$, but, taking $\rho = 2$, $\gamma = .5$, and $\lambda = 2$. We now explicitly evaluate this function.

5.1 Case I: $\sigma > 1$

Let $\beta_c^{(1)} = \sqrt{2}$, $\beta_c^{(2)} = \sigma^{-1}\sqrt{2} = \sigma^{-1}\beta_c^{(1)}$, and define

$$F^{(1)}(\beta) = \begin{cases} 1 + \beta^2/2 & \text{if } \beta \le \beta_c^{(1)} \\ \sqrt{2}\beta & \text{if } \beta \ge \beta_c^{(1)} \end{cases}$$

and

$$F^{(2)}(\beta) = \begin{cases} 1 + \beta\eta + \sigma^2\beta^2/2 & \text{if } \beta \le \beta_c^{(2)} \\ \left(\sigma\sqrt{2} + \eta\right)\beta & \text{if } \beta \ge \beta_c^{(2)} \end{cases}.$$

It follows that

$$F_{\sigma,\eta}(\beta) \quad = \quad \max\{F^{(1)}(\beta), F^{(2)}(\beta)\}.$$

If $\sigma > 1$ then $\beta_c^{(2)} = \beta_c^{(1)}/\sigma < \beta_c^{(1)}$ and

$$F_{\sigma,\eta}(\beta) = \begin{cases} \max\{1 + \beta^2/2, 1 + \beta\eta + \sigma^2\beta^2/2\} & \text{if } \beta \le \beta_c^{(2)} \\ \max\{1 + \beta^2/2, (\sigma\sqrt{2} + \eta)\beta\} & \text{if } \beta_c^{(2)} \le \beta \le \beta_c^{(1)} \\ \max\{\sqrt{2}\beta, (\sigma\sqrt{2} + \eta)\beta\} & \text{if } \beta \ge \beta_c^{(1)} \end{cases}.$$

We now evaluate the maximum. To simplify the discussion we refer to the region where $\beta \le \beta_c^{(2)}$ as Region I, the one where $\beta_c^{(2)} < \beta < \beta_c^{(1)}$ as Region II, and the one where $\beta \ge \beta_c^{(1)}$ as Region III. We also introduce the notation $a_1 = 0$, $a_2 = -(\sigma^2 - 1)/(\sigma\sqrt{2})$, and $a_3 = -\sqrt{2}(\sigma - 1)$. Observing that $a_2 = .5a_3(1 + 1/\sigma) \ge a_3$ for $\sigma > 1$, it follows that $0 = a_1 \ge a_2 \ge a_3$. Also define $\beta_c^{(3)} = -2\eta/(\sigma^2 - 1)$ and $\beta_c^{(4)} = \sigma\sqrt{2} + \eta - \sqrt{(\sigma\sqrt{2} + \eta)^2 - 2}$.

Region I. It is easy to see that, on this region, $F^{(1)}(\beta) \ge F^{(2)}(\beta)$ if and only if $\beta/2 \ge \sigma^2\beta/2 + \eta$, which holds if and only if $\beta \le -2\eta/(\sigma^2 - 1) = \beta_c^{(3)}$. Since

$\beta > 0$, this never holds for $\eta \geq 0$. Now assume that $\eta < 0$. The question is whether $-2\eta/(\sigma^2-1)$ is in Region I, thus if $-2\eta/(\sigma^2-1) \leq \sqrt{2}/\sigma$. This inequality is equivalent to $\eta \geq -(\sigma^2-1)/(\sigma\sqrt{2}) = a_2$.

Region II. If $\eta \geq 0$ then, $F^{(1)}(\beta) = 1+\beta^2/2 \leq 2 \leq (\sigma\sqrt{2}+\eta)\beta = F^{(2)}(\beta)$. If $\eta \leq a_3 = -\sqrt{2}(\sigma-1)$ then $\sigma\sqrt{2}+\eta \leq \sqrt{2}$. It follows that, in this case, we have $F^{(2)}(\beta) = (\sigma\sqrt{2}+\eta)\beta \leq \sqrt{2}\beta \leq 1+\beta^2/2 = F^{(1)}(\beta)$, where the second inequality follows from the fact that $\beta^2/2 - \sqrt{2}\beta + 1 = .5(\beta-\sqrt{2})^2 \geq 0$.

We now focus on the case $a_3 = -\sqrt{2}(\sigma-1) < \eta < 0$. We begin by introducing, for $\beta \in (-\infty,\infty)$, the function $f(\beta) = 1+\beta^2/2-(\sigma\sqrt{2}+\eta)\beta$, and noting that, for β in Region II, we have $f(\beta) = F^{(1)}(\beta) - F^{(2)}(\beta)$. To understand where $F^{(1)}$ is larger than $F^{(2)}$ we must understand where f is positive and where it is negative. First note that

$$f'(\beta) = \beta - \sigma\sqrt{2} - \eta \leq \beta - \sqrt{2} < 0 \text{ for } \beta < \sqrt{2},$$

which means that f is decreasing for $\beta < \sqrt{2}$. Since $f(0) = 1$, it follows that there is at most one point in $(0,\sqrt{2})$ at which $f(\beta) = 0$ and further, before this point f is always positive and after this point f is always negative until (at least) $\beta = \sqrt{2}$. Let us find the roots of f. We have

$$f(\beta) = \beta^2/2 - (\sigma\sqrt{2}+\eta)\beta + 1 = 0,$$

which, by the quadratic formula, holds when

$$\beta_\pm = \sigma\sqrt{2}+\eta \pm \sqrt{(\sigma\sqrt{2}+\eta)^2 - 2}.$$

Note that $\sigma\sqrt{2}+\eta > \sqrt{2}$, which guarantees that the roots are real, positive, and that $\beta_+ > \sqrt{2}$ is not in Region II. On the other hand, $\beta_- \in (0,\sqrt{2})$. Thus $f(\beta) \geq 0$ for $\beta \leq \beta_-$ and $f(\beta) \leq 0$ for $\beta \in [\beta_-,\sqrt{2})$. The question remains whether β_- is in Region II or not. It is in Region II if and only if $\beta_- > \sqrt{2}/\sigma$, which is equivalent to the condition

$$(\sigma\sqrt{2}+\eta) - \sqrt{2}/\sigma > \sqrt{(\sigma\sqrt{2}+\eta)^2 - 2}.$$

Now squaring both sides and simplifying leads to the equivalent condition

$$\eta < -\frac{\sigma^2-1}{\sigma\sqrt{2}} = a_2.$$

From here we can deduce that, on Region II, if $\eta \geq a_2$ then $F^{(1)}(\beta) \leq F^{(2)}(\beta)$, if $\eta \leq a_3$ then $F^{(1)}(\beta) \geq F^{(2)}(\beta)$, and if $\eta \in (a_3,a_2)$ then $F^{(1)}(\beta) \geq F^{(2)}(\beta)$ for $\beta \leq \beta_-$ and $F^{(1)}(\beta) \leq F^{(2)}(\beta)$ for $\beta \geq \beta_-$. Finally, we note that $\beta_- = \beta_c^{(4)}$.

Region III. Note that here $F^{(1)}(\beta) \geq F^{(2)}(\beta)$ if and only if $\sqrt{2} \geq \sigma\sqrt{2}+\eta$, which holds if and only if $\eta \leq (1-\sigma)\sqrt{2} = a_3$.

We now summarize the above as follows. If $\eta \geq a_1 = 0$ then $F_{\sigma,\eta}(\beta) = F^{(2)}(\beta)$. If $-(\sigma^2-1)/(\sigma\sqrt{2}) = a_2 \leq \eta < a_1 = 0$ then $F_{\sigma,a}(\beta) = F^{(3)}(\beta)$, where

$$F^{(3)}(\beta) = \begin{cases} 1+\beta^2/2 & \text{if } \beta \leq \beta_c^{(3)} \\ 1+\beta\eta+\sigma^2\beta^2/2 & \text{if } \beta_c^{(3)} \leq \beta \leq \beta_c^{(2)} \\ (\sigma\sqrt{2}+\eta)\beta & \text{if } \beta \geq \beta_c^{(2)} \end{cases} .$$

If $-\sqrt{2}(\sigma-1) = a_3 < \eta < a_2 = -(\sigma^2-1)/(\sigma\sqrt{2})$ then $F_{\sigma,\eta}(\beta) = F^{(4)}(\beta)$, where

$$F^{(4)}(\beta) = \begin{cases} 1+\beta^2/2 & \text{if } \beta \leq \beta_c^{(4)} \\ (\sigma\sqrt{2}+\eta)\beta & \text{if } \beta \geq \beta_c^{(4)} \end{cases} .$$

If $\eta \leq a_3 = -\sqrt{2}(\sigma-1)$ then $F_{\sigma,\eta}(\beta) = F^{(1)}(\beta)$.

It is straightforward to see that for all fixed $\sigma > 1$ and $\eta \in \mathbb{R}$, the function $F_{\sigma,\eta}(\beta)$ is continuous in β. However, when $a_3 \leq \eta < a_2$ the function $F_{\sigma,\eta}(\beta) = F^{(4)}(\beta)$ is not differentiable in β at $\beta = \beta_c^{(4)}$ thus it has a first order phase transition there. Similarly, when $a_2 \leq \eta < a_1$ the function $F_{\sigma,a}(\beta) = F^{(3)}(\beta)$ is not differentiable in β at $\beta = \beta_c^{(3)}$, and it has a first order phase transition there. On the other hand, on this region, it is differentiable at $\beta = \beta_c^{(2)}$, but only once, thus it has a second order phase transition at that point.

5.2 Case II: $\sigma \leq 1$

We begin with the case $\sigma = 1$. In this case, $\beta_c^{(2)} = \beta_c^{(1)}/\sigma = \beta_c^{(1)}$ and

$$F_{1,\eta}(\beta) = \begin{cases} \max\{1+\beta^2/2, 1+\beta\eta+\beta^2/2\} & \text{if } \beta \leq \beta_c^{(1)} \\ \max\{\sqrt{2}\beta, (\sqrt{2}+\eta)\beta\} & \text{if } \beta \geq \beta_c^{(1)} \end{cases} .$$

It follows that, if $\eta \geq 0$ then $F_{1,\eta}(\beta) = F^{(2)}(\beta)$ and if $\eta \leq 0$ then $F_{1,\eta}(\beta) = F^{(1)}(\beta)$.

We will not deal with the case where $\sigma < 1$ directly. Instead, we will show that results for this case follow directly from results for the case $\sigma > 1$. First note that, for any $\sigma > 0$, we have

$$F_{\sigma,\eta}(\beta) = \max\{F^{(1)}(\beta), F^{(2)}(\beta)\} = \max\{F^{(1)}(\beta), F^{(1)}(\sigma\beta)+\eta\beta\}.$$

It follows that

$$F_{\sigma,\eta}(\beta) = F_{1/\sigma,-\eta/\sigma}(\beta\sigma)+\eta\beta.$$

From here, we can easily use the results for the case $\sigma > 1$ to get results for this case.

If $\eta \leq 0$ then $F_{\sigma,\eta}(\beta) = F^{(1)}(\beta)$. If $0 < \eta \leq (1-\sigma^2)/\sqrt{2}$ then $F_{\sigma,\eta}(\beta) = F^{(5)}(\beta)$, where

$$F^{(5)}(\beta) = \begin{cases} 1 + \eta\beta + \sigma^2\beta^2/2 & \text{if } \beta \leq \beta_c^{(3)} \\ 1 + \beta^2/2 & \text{if } \beta_c^{(3)} \leq \beta \leq \beta_c^{(1)} \\ \sqrt{2}\beta & \text{if } \beta \geq \beta_c^{(1)} \end{cases},$$

where, as before, $\beta_c^{(1)} = \sqrt{2}$ and $\beta_c^{(3)} = 2\eta/(1-\sigma^2)$. If $(1-\sigma^2)/\sqrt{2} < \eta < \sqrt{2}(1-\sigma)$ then $F_{\sigma,\eta}(\beta) = F^{(6)}(\beta)$, where

$$F^{(6)}(\beta) = \begin{cases} 1 + \eta\beta + \sigma^2\beta^2/2 & \text{if } \beta \leq \beta_c^{(5)} \\ \sqrt{2}\beta & \text{if } \beta \geq \beta_c^{(5)} \end{cases},$$

where $\beta_c^{(5)} = \left(\sqrt{2} - \eta - \sqrt{(\sqrt{2}-\eta)^2 - 2\sigma^2}\right)/\sigma^2$. Finally, if $\eta \geq \sqrt{2}(1-\sigma)$ then $F_{\sigma,\eta}(\beta) = F^{(2)}(\beta)$.

Note that, as in the case where $\sigma > 1$, the function $F_{\sigma,\eta}$ is always continuous in β. Further, when $0 < \eta \leq (1-\sigma^2)/\sqrt{2}$ we have a first order phase transition at $\beta = \beta_c^{(3)}$ and a second order one at $\beta = \beta_c^{(1)}$. When $(1-\sigma^2)/\sqrt{2} < \eta < \sqrt{2}(1-\sigma)$ we have a first order phase transition at $\beta = \beta_c^{(5)}$.

5.3 Phase Diagram

In Figure 1 we plot the phase diagram in a coordinate system plotting σ against η. The solid line represents the line $\eta = 0$, the dashed line represents the line $\eta = \sqrt{2}(1-\sigma)$, and the dotted line represents the curve $\eta = \frac{1-\sigma^2}{\sqrt{2}(\sigma \vee 1)}$, where $(\sigma \vee 1)$ represents the maximum of σ and 1. These curves divide the right half-plane into six regions, labeled $1, 2, \ldots, 6$. On Region i, $F_{\sigma,\eta}(\beta) = F^{(i)}(\beta)$ for $i = 1, 2, \ldots, 6$.

A Proofs for Section 3

Proof of Lemma 2. Let $\phi(x) = (2\pi)^{-1/2}e^{-x^2/2}$ and $\Phi(x) = \int_{-\infty}^{x}\phi(t)\mathrm{d}t$ be, respectively, the pdf and cdf of the $N(0,1)$ distribution. It is well-known, see e.g. [12], that for $x > 0$

$$\frac{x}{x^2+1}\phi(x) \leq 1 - \Phi(x) \leq \frac{1}{x}\phi(x),$$

and thus

$$-\ln(x + 1/x) - .5\ln(2\pi) - x^2/2 \leq \ln(1 - \Phi(x)) \leq -\ln x - .5\ln(2\pi) - x^2/2.$$

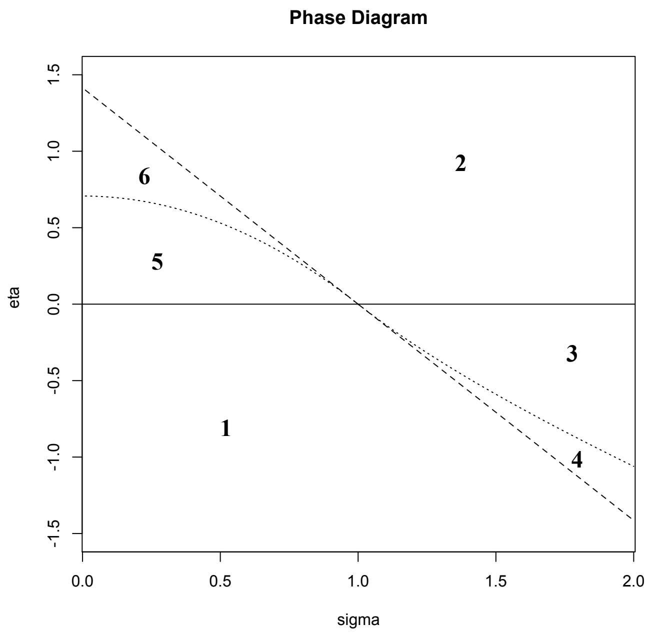

Figure 1: Phase Diagram. On Region i, $F_{\sigma,\eta}(\beta) = F^{(i)}(\beta)$ for $i = 1, 2, \ldots, 6$.

It follows that

$$1 - \Phi(x) \sim \frac{1}{x}\phi(x) \text{ as } x \to \infty$$

and

$$h(x) = -\ln(1 - \Phi(x)) \sim x^2/2 \text{ as } x \to \infty.$$

Thus $h \in R_2$ and

$$\frac{xh'(x)}{h(x)} = \frac{x\phi(x)}{-\ln(1-\Phi(x))(1-\Phi(x))} \sim \frac{x^2}{x^2/2} = 2.$$

From here, applying Lemma 1 shows that $h \in NR_2$. The formula for H follows immediately from the fact that, if $X \sim N(0,1)$, then $\mathrm{E}\left[e^{tX}\right] = e^{.5t^2}$. $\square$

For simplicity of notation we write

$$S_t = S_t(1) = \sum_{i=1}^{M_t} e^{t^{1/\rho}X_i}, \qquad U_t = U_t(1) = \sum_{i=1}^{R(t)} e^{t^{1/\rho}X_i},$$

and

$$\lambda^* = \rho - 1.$$

We now collect several results from [2].

Proposition 2. *In all of the following, take $\kappa = 1$.*
1. If $\lambda > \lambda^*$ then

$$\frac{U_t}{R(t)e^{H(t^{1/\rho})}} \xrightarrow{p} \kappa. \tag{12}$$

2. If $\lambda < \lambda^*$ then there is a deterministic sequence $A_\rho(t)$ with

$$\frac{U_t}{A_\rho(t^{1/\rho})} \xrightarrow{d} \kappa^{1/\alpha}\xi_\alpha, \tag{13}$$

where ξ_α is a fully right-skewed α-stable random variable with $\alpha = (\lambda/\lambda^*)^{1/\rho}$ and characteristic function

$$\phi_\alpha(t) = \exp\left\{-\Gamma(1-\alpha)|t|^\alpha e^{-.5i\pi\alpha \mathrm{sgn}(t)}\right\}. \tag{14}$$

Further,

$$\ln A_\rho(t) \sim \rho(\lambda/\lambda^*)^{1-1/\rho}\gamma t^\rho.$$

3. If $\lambda = \lambda^*$ then there is a deterministic sequence $B_\rho(t)$ with

$$\frac{U_t}{R(t)B_\rho(t^{1/\rho})} \xrightarrow{p} \kappa. \tag{15}$$

Further, $B_\rho(t) \le e^{H(t)}$ and there are functions H_0 and b_ρ such that $B_\rho(t) = e^{H_0(t)}b_\rho(t)$ with $H_0(t) \sim H(t)$ and $b_\rho(t) \to \infty$.

Proof. Combining Theorem 2.1, Theorem 2.3, Theorem 2.4, and (5.21) from [2] gives everything except the representation of $B_\rho(t)$, which follows from (5.32), (5.23), (5.27), (5.19), and Lemma 5.6 in [2] along with the fact that $\lambda = \rho - 1$ in this case. □

We will need a randomized version of these result. Toward this end we recall a transfer theorem, which can be found in, e.g. Theorem 4.1.2 of [11].

Proposition 3. *Let $\{X_{t,n}\}$ be random variables such that, for a fixed t, $X_{t,1}, X_{t,2}, \dots$ are iid, and let K_t be a random variable independent of $\{X_{t,n}\}$. Assume that there are natural numbers k_t with $k_t \to \infty$ as $t \to \infty$ such that*

$$\sum_{n=1}^{k_t} X_{t,n} \xrightarrow{d} Y \text{ as } t \to \infty,$$

where Y has characteristic function ϕ. If

$$\frac{K_t}{k_t} \xrightarrow{p} c \in [0, \infty)$$

then

$$\sum_{n=1}^{K_t} X_{t,n} \xrightarrow{d} Y^*,$$

where Y^* has characteristic function ϕ^* with $\phi^*(t) = (\phi(t))^c$.

Combining this with Proposition 2 immediately gives the following.

Proposition 4. *If*

$$\frac{M_t}{R(t)} \xrightarrow{p} c \in (0,\infty),$$

then the results of Proposition 2 remain true with U_t replaced by S_t and κ replaced by c.

We can now prove a result, from which Theorem 1 follows immediately.

Proposition 5. *If $\lambda \in (0,\infty)$ then*

$$\frac{\ln S_t(\beta)}{t} \xrightarrow{p} \gamma\beta^\rho L_\rho(\lambda/\beta^\rho),$$

where

$$L_\rho(\lambda) = \begin{cases} \lambda+1 & \text{if } \lambda \geq \lambda^* \\ \rho\,(\lambda/\lambda^*)^{1-1/\rho} & \text{if } \lambda \leq \lambda^* \end{cases}.$$

The proof is similar to that of Theorem 9.1 in [1]. We include it here for completeness.

Proof. In this proof, all limits should be understood in the sense of convergence in probability. We begin with the case $\beta = 1$. If $\lambda > \lambda^*$ then, by Slutsky's Theorem, the continuous mapping theorem, Proposition 4, (5), and (6) it follows that

$$\lim_{t\to\infty} \frac{\ln S_t}{t} = \lim_{t\to\infty} \frac{\ln\left(\frac{S_t}{R(t)e^{H(t^{1/\rho})}}\right) + \ln R(t) + H(t^{1/\rho})}{t} = \gamma(\lambda+1).$$

Similarly, if $\lambda < \lambda^*$ then

$$\begin{aligned} \lim_{t\to\infty} \frac{\ln S_t}{t} &= \lim_{t\to\infty} \frac{\ln\left(\frac{S_t}{A_\rho(t^{1/\rho})}\right) + \ln A_\rho(t^{1/\rho})}{t} \\ &= \lim_{t\to\infty} \frac{\ln\left(\frac{S_t}{A_\rho(t^{1/\rho})}\right) + \rho(\lambda/\lambda^*)^{1-1/\rho}\gamma t}{t} = \gamma\rho(\lambda/\lambda^*)^{1-1/\rho}. \end{aligned}$$

Finally, if $\lambda = \lambda^*$ then

$$\begin{aligned} \limsup_{t\to\infty} \frac{\ln S_t}{t} &= \limsup_{t\to\infty} \frac{\ln\left(\frac{S_t}{R(t)B_\rho(t^{1/\rho})}\right) + \ln R(t) + \ln B_\rho(t^{1/\rho})}{t} \\ &\leq \limsup_{t\to\infty} \frac{\ln\left(\frac{S_t}{R(t)B_\rho(t^{1/\rho})}\right) + \ln R(t) + H(t^{1/\rho})}{t} = \gamma(\lambda+1), \end{aligned}$$

while

$$\begin{aligned}
\liminf_{t\to\infty}\frac{\ln S_t}{t} &= \liminf_{t\to\infty}\frac{\ln\left(\frac{S_t}{R(t)B_\rho(t^{1/\rho})}\right)+\ln R(t)+\ln B_\rho(t^{1/\rho})}{t}\\
&= \liminf_{t\to\infty}\frac{\ln\left(\frac{S_t}{R(t)B_\rho(t^{1/\rho})}\right)+\ln R(t)+H_0(t^{1/\rho})+\ln b_\rho(t^{1/\rho})}{t}\\
&\geq \liminf_{t\to\infty}\frac{\ln\left(\frac{S_t}{R(t)B_\rho(t^{1/\rho})}\right)+\ln R(t)+\frac{H_0(t^{1/\rho})}{H(t^{1/\rho})}H(t^{1/\rho})}{t}\\
&= \gamma(\lambda+1).
\end{aligned}$$

We now extend to the case $\beta \neq 1$. Since (5) implies that

$$\ln \mathrm{E}\left[e^{t\beta X}\right] \sim \gamma\beta^\rho t^\rho \text{ as } t\to\infty,$$

it follows that we must replace γ with $\gamma\beta^\rho$ and λ with λ/β^ρ. □

B Proofs for Section 4

Lemma 3. *If $a,\gamma>0$, $g\in NR_{1/\gamma}$, and $g(x)\sim ax^{1/\gamma}$ as $x\to\infty$, then*

$$g^{-1}(x)\sim(x/a)^\gamma.$$

Proof. Since $g^{-1}(x)\to\infty$, change of variables implies that

$$\begin{aligned}
\lim_{x\to\infty}\frac{g^{-1}(x)}{(x/a)^\gamma} &= \lim_{x\to\infty}\frac{g^{-1}(x)}{(g(g^{-1}(x))/a)^\gamma} = \lim_{x\to\infty}\frac{x}{(g(x)/a)^\gamma}\\
&= \lim_{x\to\infty}\frac{x}{x\left(\frac{g(x)}{ax^{1/\gamma}}\right)^\gamma}=1,
\end{aligned}$$

as required. □

For simplicity of notation, we write

$$S_t=\sum_{i=1}^{M_t}e^{u(t)X_i} \text{ and } U_t=\sum_{i=1}^{R(t)}e^{u(t)X_i}.$$

The following proposition collects several results from [4].

Proposition 6. *In all of the following, take $\kappa=1$.*
1. If $\lambda>1$ then

$$\frac{U_t}{R(t)e^{H(u(t))}}\xrightarrow{p}\kappa. \tag{16}$$

2. If $\lambda < 1$ then there is a deterministic sequence $A(t)$ with

$$\frac{U_t}{A(u(t))} \xrightarrow{d} \kappa^{1/\alpha}\xi_\alpha, \tag{17}$$

where ξ_α is a fully right-skewed α-stable random variable with $\alpha = \lambda$ and characteristic function is given by ϕ_α as in (14) and

$$\ln A(t) \sim \gamma t \ln t.$$

3. If $\lambda = 1$ then there is a deterministic sequence $B(t)$ with

$$\frac{U_t}{R(t)B(u(t))} \xrightarrow{p} \kappa. \tag{18}$$

Further, $B(t) \le e^{H(t)}$ and $B(t) = A(t)e^{-\gamma t}b(t)$ for some function b with $b(t) \to \infty$.

Proof. This combines Theorem 2.1, Theorem 2.4, Theorem 2.5, Lemma 6.3, and (5.4) in [4]. □

Combining this with Proposition 3 gives the following.

Proposition 7. *If*

$$\frac{M_t}{R(t)} \xrightarrow{p} c \in (0, \infty),$$

then the results of Proposition 6 remain true with U_t replaced by S_t and κ replaced by c.

We can now prove Theorem 2.

Proof of Theorem 2. In this proof all limits should be understood in the sense of convergence in probability. We begin with the case $\beta = 1$. If $\lambda > 1$ then, by Slutsky's Theorem, the continuous mapping theorem, Proposition 7, (9), (11), and (10) it follows that

$$\begin{aligned}
\lim_{t\to\infty} \frac{\log S_t}{t} &= \lim_{t\to\infty} \frac{\log\left(\frac{S_t}{R(t)e^{H(u(t))}}\right) + \log R(t) + H(u(t))}{t} \\
&= \lim_{t\to\infty} \frac{\ln R(t\ln t) + H(t)}{t \ln t} \\
&= \lim_{t\to\infty} \frac{\ln R(t\ln t)}{t\ln t} + \frac{\frac{H(t)}{t} - \ln\left(g^{-1}(t)\right) + \ln\left(g^{-1}(t)\right)}{\ln t} \\
&= \frac{\ln\left(\frac{g^{-1}(t)}{(t/a)^\gamma}\right) + \gamma \ln t - \gamma \ln a}{\ln t} = \gamma.
\end{aligned}$$

Similarly, if $\lambda < 1$ then

$$\begin{aligned}
\lim_{t\to\infty} \frac{\ln S_t}{t} &= \lim_{t\to\infty} \frac{\log\left(\frac{S_t}{A(u(t))}\right) + \ln A(u(t))}{t} \\
&= \lim_{t\to\infty} \frac{\ln A(t)}{t\ln t} = \gamma.
\end{aligned}$$

Finally, if $\lambda = 1$ then

$$\begin{aligned}
\limsup_{t\to\infty} \frac{\log S_t}{t} &= \limsup_{t\to\infty} \frac{\ln\left(\frac{S_t}{R(t)B(u(t))}\right) + \ln R(t) + \log B(u(t))}{t} \\
&\le \limsup_{t\to\infty} \frac{\ln R(t\ln t) + H(t)}{t\ln t} = \gamma
\end{aligned}$$

while

$$\begin{aligned}
\liminf_{t\to\infty} \frac{\ln S_t}{t} &= \liminf_{t\to\infty} \frac{\log\left(\frac{S_t}{R(t)B(u(t))}\right) + \ln R(t) + \ln B(u(t))}{t} \\
&= \liminf_{t\to\infty} \frac{\ln R(t\ln t) + \log B(t)}{t\ln t} \\
&\ge \liminf_{t\to\infty} \frac{\ln R(t\ln t) + \log A(t) - \gamma t}{t\ln t} \\
&= \liminf_{t\to\infty} \frac{\ln A(t)}{t\ln t} = \gamma.
\end{aligned}$$

We now extend to the case $\beta \neq 1$. Since

$$-\log P(\beta X > x) = h(x/\beta),$$

we must replace γ by $\gamma\beta$ and λ by λ/β. □

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