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A RANDOM VARIABLE THAT DOES NOT BELONG TO A DOMAIN OF ATTRACTION, BUT ITS ABSOLUTE VALUE DOES

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Abstract

In this paper we give an example of a distribution ν_α, which does not belong to the domain of attraction of any stable distribution. However, if X has distribution ν_α, then the distribution of $|X|$ belongs to the domain of attraction of an α-stable distribution. The distribution has a simple structure, and may be useful for pedagogical purposes.

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The central limit theorem is one of the best known and most celebrated results in probability theory. In its simplest form, it says that if $\{X_i\}$ is a sequence of independent and identically distributed (i.i.d.) random variables with a finite variance and not concentrated at a point, then there exist deterministic sequences $\{a_n\}$ and $\{b_n\}$ such that

$$a_n \sum_{i=1}^{n} X_i + b_n \xrightarrow{\text{D}} Z, \tag{1}$$

where Z has a standard normal distribution and '$\xrightarrow{\text{D}}$' denotes convergence in distribution. (We say that a random variable X, or equivalently its distribution ν, is concentrated at a point if $\mathbb{P}(X = c) = 1$ for some $c \in \mathbb{R}$.) Somewhat less well known is what happens when the variance is infinite. In this case, one of the following may happen:

(i) (1) still holds with Z having a standard normal distribution,

(ii) (1) holds with Z having an infinite variance stable distribution,

(iii) (1) does not hold for any Z that is not concentrated at a point.

Before proceeding, we recall the definition of a stable distribution. Let μ be a probability distribution not concentrated at a point and let $\{Z_i\}$ be a sequence of i.i.d. random variables with distribution μ. We say that μ is stable if for any n there exist nonrandom constants $c_n > 0$ and $d_n \in \mathbb{R}$ with

$$c_n(Z_1 + Z_2 + \cdots + Z_n) + d_n \stackrel{\text{D}}{=} Z_1,$$

where '$\stackrel{\text{D}}{=}$' denotes equality in distribution. In this case, we necessarily have $c_n = n^{-1/\alpha}$ for some $\alpha \in (0, 2]$. We call the corresponding distribution α-stable. A distribution is 2-stable if and only if it is a normal distribution. If $\alpha \in (0, 2)$, then the distribution has an infinite variance. We call such distributions *infinite variance stable distributions*.

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The domain of attraction (DOA) of an α-stable distribution μ is the collection of all distributions ν such that, if $\{X_i\}$ is a sequence of i.i.d. random variables with distribution ν, then (1) holds with Z having distribution μ. We similarly say that a random variable X belongs to the DOA of μ if the distribution of X does. Before characterizing when a distribution belongs to the DOA of some α-stable distribution, we recall the following definition. For $\beta \in \mathbb{R}$, a function $f\colon [0,\infty) \mapsto [0,\infty)$ is said to regularly varying with index β if

$$\lim_{t\to\infty} \frac{f(xt)}{f(t)} = x^{\beta}.$$

In this case, we write that $f \in \mathrm{RV}_{\beta}$. It is straightforward to check that, if there exists a $c > 0$ with $f(t)t^{-\beta} \to c$ as $t \to \infty$, then $f \in \mathrm{RV}_{\beta}$. The following result is well known (see e.g. [3, p. 577]).

Result 1. *Fix $\alpha \in (0, 2]$, let ν be a probability distribution, let X be a random variable with distribution ν, and let*

$$\psi(x) = \mathbb{E}[X^2\,\mathbf{1}_{|X|\le x}],$$

where '$\mathbf{1}$' denotes the indicator function. A necessary condition for ν to belong to the DOA of some α-stable distribution is

$$\psi \in \mathrm{RV}_{2-\alpha}\,. \tag{2}$$

If $\alpha = 2$ this is also sufficient. If $\alpha \in (0, 2)$ and (2) holds, then a necessary and sufficient condition for ν to belong to the DOA of some α-stable distribution is that there exists a $p \in [0, 1]$ with

$$\lim_{x\to\infty} \frac{\mathbb{P}(X > x)}{\mathbb{P}(|X| > x)} = p \quad \textit{and} \quad \lim_{x\to\infty} \frac{\mathbb{P}(X < -x)}{\mathbb{P}(|X| > x)} = 1 - p. \tag{3}$$

The condition in (3) is called the *tail balance condition*. To better understand when these conditions hold, we give several examples.

Example 1. A Pareto distribution with parameters $\alpha, b > 0$ has a probability density function (PDF) given by

$$f(t) = \alpha b^{\alpha} t^{-\alpha-1}, \qquad t > b.$$

It is easy to check that when $\alpha \in (0, 2)$ this distribution belongs to the DOA of an α-stable distribution, and when $\alpha \ge 2$ it belongs to the DOA of a 2-stable distribution, i.e. the normal distribution. Note that, when $\alpha = 2$ the distribution has an infinite variance, but still belongs to the DOA of the normal distribution.

Example 2. A symmetric Pareto distribution with parameters $\alpha, b > 0$ has a PDF given by

$$f(t) = \frac{\alpha}{2} b^{\alpha} |t|^{-\alpha-1}, \qquad |t| > b.$$

Again, it is easy to check that when $\alpha \in (0, 2)$ this distribution belongs to the DOA of an α-stable distribution, and when $\alpha \ge 2$ it belongs to the DOA of a 2-stable distribution. For more on this distribution, see [4].

Example 3. One of the most famous distributions that does not belong to the DOA of any stable distribution is provided by the St. Petersburg game. In this game, the amount won by a player is a random variable X with

$$\mathbb{P}(X = 2^k) = 2^{-k}, \qquad k = 1, 2, \ldots.$$

It can be shown that (2) does not hold in this case for any α. For more about this distribution and its extensions, see [1], [2], and the references therein.

The main goal of this paper is to give an example of a probability distribution ν_α such that, if X has distribution ν_α, then

(i) X does not belong to the DOA of any stable distribution, but

(ii) $Y = |X|$ belongs to the DOA of some α-stable distribution.

It is not difficult to see that such a distribution must satisfy (2), but must fail to satisfy (3). While the existence of such distributions is well known, we have not seen an explicit example in the literature. Further, our example has a simple structure, which clearly illustrates how (2) holds, but (3) fails. As such, it may be useful for pedagogical purposes. The example is as follows.

Fix $\alpha \in (0, 2)$ and consider a distribution ν_α with PDF

$$f_\alpha(t) = \begin{cases} \dfrac{\alpha}{4}\mathrm{e}^{-\pi}\dfrac{2 - \cos(\log(|t|^{-\alpha})) - \sin(\log(|t|^{-\alpha}))}{|t|^{\alpha+1}}, & t < -\mathrm{e}^{-\pi/\alpha}, \\ \dfrac{\alpha}{4}\mathrm{e}^{-\pi}\dfrac{2 + \cos(\log(t^{-\alpha})) + \sin(\log(t^{-\alpha}))}{t^{\alpha+1}}, & t > \mathrm{e}^{-\pi/\alpha}, \\ 0, & \text{otherwise.} \end{cases}$$

Note that this is essentially a symmetric Pareto distribution, as defined in Example 2, but the tails have been modified to make them less smooth. This lack of smoothness will make (3) fail.

We begin by showing that ν_α satisfies (2). Let X have distribution ν_α and note that, for $x > \mathrm{e}^{-\pi/\alpha}$,

$$\begin{aligned} \psi(x) &= \mathrm{E}[|X|^2 \mathbf{1}_{[|X| \le x]}] \\ &= \alpha \mathrm{e}^{-\pi} \int_{\mathrm{e}^{-\pi/\alpha}}^{x} t^{1-\alpha}\, \mathrm{d}t \\ &= \frac{\alpha}{2-\alpha}\mathrm{e}^{-\pi}(x^{2-\alpha} - \mathrm{e}^{-\pi(2/\alpha - 1)}). \end{aligned}$$

It follows that

$$\lim_{x\to\infty} \psi(x) x^{-(2-\alpha)} = \frac{\alpha}{2-\alpha}\mathrm{e}^{-\pi} > 0;$$

hence, $\psi \in \mathrm{RV}_{2-\alpha}$. To see that (3) does not hold, note that, for $x > \mathrm{e}^{-\pi/\alpha}$,

$$\mathbb{P}(|X| > x) = \alpha \mathrm{e}^{-\pi} \int_x^\infty t^{-\alpha-1}\, \mathrm{d}t = \mathrm{e}^{-\pi} x^{-\alpha}$$

and

$$\begin{aligned} \mathbb{P}(X > x) &= \frac{\alpha}{4}\mathrm{e}^{-\pi} \int_x^\infty \frac{2 + \cos(\log(t^{-\alpha})) + \sin(\log(t^{-\alpha}))}{t^{\alpha+1}}\, \mathrm{d}t \\ &= \frac{1}{4}\mathrm{e}^{-\pi} \int_0^{x^{-\alpha}} (2 + \cos(\log u) + \sin(\log u))\, \mathrm{d}u \\ &= \frac{1}{4}\mathrm{e}^{-\pi}(2 - \sin(\alpha \log x)) x^{-\alpha}, \end{aligned}$$

where the second line follows by the substitution $u = t^{-\alpha}$ and the third by the facts that

$$\frac{\mathrm{d}}{\mathrm{d}u} u \sin(\log u) = \sin(\log u) + \cos(\log u)$$

and

$$\lim_{u \to 0^+} u \sin(\log u) = 0.$$

It follows that

$$\frac{\mathbb{P}(X > x)}{\mathbb{P}(|X| > x)} = \frac{1}{4}(2 - \sin(\alpha \log x)).$$

It is easy to see that this has no limit as $x \to \infty$. Specifically, consider the sequences $s_n = \mathrm{e}^{\pi n/\alpha}$ and $t_n = \mathrm{e}^{\pi(2n+0.5)/\alpha}$. It follows that

$$\lim_{n \to \infty} \frac{\mathbb{P}(X > s_n)}{\mathbb{P}(|X| > s_n)} = \lim_{n \to \infty} \frac{1}{4}(2 - \sin(\pi n)) = 0.5$$

and

$$\lim_{n \to \infty} \frac{\mathbb{P}(X > t_n)}{\mathbb{P}(|X| > t_n)} = \lim_{n \to \infty} \frac{1}{4}(2 - \sin(\pi(2n + 0.5))) = 0.25.$$

We can similarly show that $\mathbb{P}(X < -x)/\mathbb{P}(|X| > x)$ has no limit as $x \to \infty$.

The above shows that ν_α (and equivalently X) does not belong to the DOA of any stable distribution. Now, set $Y = |X|$. It is not difficult to check that the PDF of Y is given by

$$g_\alpha(t) = \alpha \mathrm{e}^{-\pi} t^{-\alpha-1}, \qquad t > \mathrm{e}^{-\pi/\alpha},$$

which is a Pareto distribution, as defined in Example 1. Thus, when $\alpha \in (0, 2)$ it belongs to the DOA of an α-stable distribution.

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