CHAPTER

Techniques of Circuit Analysis

4

Up to this point we have been able to analyze relatively simple resistive circuits by the intelligent application of Kirchhoff's laws in combination with Ohm's law. This approach can be used for all circuits, but as the circuits become structurally more complicated and involve more and more elements, we will soon find this direct method quite cumbersome. Our purpose in this chapter is to introduce two powerful techniques of circuit analysis that facilitate the analysis of complex circuit structures: the node-voltage method and the mesh-current method. In addition to these two general methods of analysis, we will also discuss some additional techniques for simplifying circuits. We have already seen how series-parallel reductions and delta-to-wye transformations can be used to simplify a given structure. We will now add source transformations and Thévenin-Norton equivalent circuits to our list of simplification techniques.

SPICE SPICE programs for analyzing dc circuits: Secs. 5 and 6 (manual pp. 12 and 16)

Before we begin our discussion of the node-voltage and mesh-current methods of circuit analysis, let us pause and reflect on the groundwork that has been laid to this point. In Chapter 1 we introduced current and voltage as the two variables used to describe the behavior of the basic circuit elements. In Chapter 2 we discussed voltage and current sources along with the circuit parameter of resistance. These three types of basic circuit elements were chosen to start our discussion of circuit analysis because all the basic analytical techniques can be explored using interconnections of just these elements. In both Chapters 2 and 3 we introduced circuit analysis through the direct application of

Ohm's law and Kirchhoff's laws. Ohm's law is critical because it describes the relationship between the current and voltage at the terminals of a resistor, and Kirchhoff's laws are important because they describe the constraints imposed on currents and voltages due to the interconnections of the basic elements. We also noted in Section 2.5 in our discussion of both the flashlight circuit and the dependent-source circuit that we can simplify the analysis problem by first concentrating on finding the element currents. We are now ready to introduce the node-voltage and mesh-current methods of circuit analysis. Keep in mind that these two methods are of interest because they give us two systematic methods for describing circuits with the minimum number of simultaneous equations.

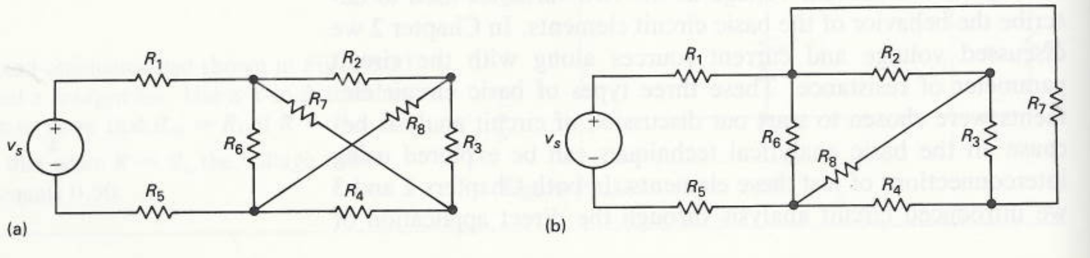
One final comment: In this chapter we will emphasize the *mechanics* of implementing the node-voltage and mesh-current methods. In Chapter 5 we will show why either method leads to a set of independent simultaneous equations.

4.1 TERMINOLOGY

In order to discuss the more elegant methods of circuit analysis, we must define a few basic terms that will enable us to give a clear, concise description of the important features of a given circuit. Thus far all of our circuits have been *planar circuits*—that is, those circuits that can be drawn on a plane such that no branches cross over each other. A circuit that can be drawn with branches crossing over each other is still considered planar if it can be redrawn with no crossover branches. For example, the circuit shown in Fig. 4.1(a) is a planar circuit since it can be redrawn as shown in Fig. 4.1(b). An example of a nonplanar circuit is shown in Fig. 4.2.

The node-voltage method can be applied to both planar and nonplanar circuits, whereas the mesh-current method is limited to planar circuits. For nonplanar circuits, the mesh-current method is replaced by a technique known as the loop-current method. We will discuss the loop-current method in Chapter 5.

Figure 4.1 (a) A planar circuit and (b) the same circuit redrawn to verify that it is planar.



Describing a Circuit—The Vocabulary

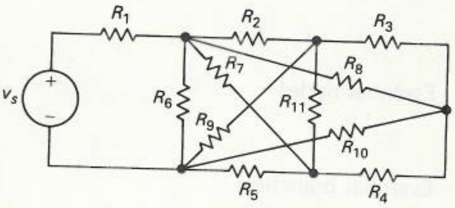


Figure 4.2 A nonplanar circuit.

We have already defined (Section 1.4) what we mean by an ideal basic circuit element. When the basic circuit elements are interconnected to form a circuit, the resulting interconnection is described in terms of nodes, paths, branches, loops, and meshes. We defined both a node and a closed path, or loop, in Section 2.4. Our purpose here is to review those definitions and then expand our vocabulary to include the terms "path," "branch," and "mesh."

Node

A *node* is a point in a circuit where two or more circuit elements join.

Path

A *path* is formed whenever a set of adjoining basic circuit elements is traced, in order, without passing through a connecting node more than once.

Loop

A *closed path,* or *loop,* is created by starting at a selected node and then tracing through a set of connected basic circuit elements in such a manner that we return to the original starting node without passing through any intermediate node more than once.

Branch

A *branch* is a path that connects two nodes.

Mesh

A *mesh* is a special type of loop; that is, it does not contain any other loops within it.

These characteristics of a circuit are illustrated with reference to the circuit shown in Fig. 4.3, a careful study of which will reveal that there are

1. seven nodes (a, b, c, d, e, f, and g);
2. ten branches (v_1, R_1, R_2, R_3, v_2, R_4, R_5, R_6, R_7, and I); and
3. four meshes (v_1–R_1–R_5–R_3–R_2, v_2–R_2–R_3–R_6–R_4, R_5–R_7–R_6, and R_7–I).

Also note that there are several loops that are not meshes. For example, v_1–R_1–R_5–R_6–R_4–v_2 forms a closed loop, but this is not a mesh because other closed loops can be found within it. A path that is neither closed nor a branch is the trace v_2–R_2–R_3–R_5–R_7.

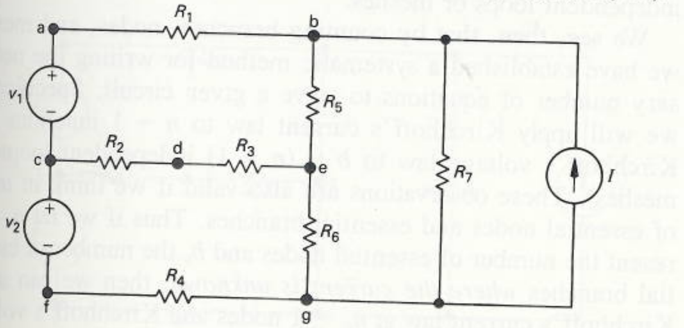


Figure 4.3 A circuit illustrating nodes, branches, meshes, paths, and loops.

Essential nodes

Essential branches

We will find in the work that follows that it is often convenient to identify only those nodes in the circuit that join *three or more* elements. We will refer to such nodes as *essential nodes*. We also find it convenient to identify only those paths that connect essential nodes *without passing through an essential node*. We will refer to such paths as *essential branches*. In the circuit shown in Fig. 4.3 there are four essential nodes (b, c, e, and g) and seven essential branches (v_1–R_1; R_2–R_3; v_2–R_4; R_5; R_6; R_7; and I). Note that in general the number of essential nodes will be less than or equal to the number of nodes and the number of essential branches will be less than or equal to the number of branches.

Simultaneous Equations—How Many?

Why are we interested in noting how many nodes, branches, and meshes a given circuit has? It turns out that these attributes of the circuit tell us how many simultaneous equations we must derive in order to solve the circuit. This can be explained as follows. The number of unknown currents in the circuit will equal the number of branches *where the current is not known*. We let b represent this number. For example, in the circuit shown in Fig. 4.3 there are nine branches where the current is unknown. We know from elementary algebra that we must have b independent equations in order to solve a circuit with b unknown currents. If we let n represent the number of nodes in the circuit, we know that we can derive $n - 1$ independent equations by applying Kirchhoff's current law to any set of $n - 1$ nodes. (The application of the current law to the nth node would not generate an independent equation since this equation can be derived from the previous $n - 1$ equations. See Drill Exercise 4.2.) Knowing that we need b equations to describe a given circuit and knowing further that we can obtain $n - 1$ of these equations from Kirchhoff's current law, it follows that the remaining $b - (n - 1)$ equations must come from applying Kirchhoff's voltage law to independent loops or meshes.

Independent equations

We see, then, that by counting branches, nodes, and meshes we have established a systematic method for writing the necessary number of equations to solve a given circuit. Specifically, we will apply Kirchhoff's current law to $n - 1$ junctions and Kirchhoff's voltage law to $b - (n - 1)$ independent loops (or meshes). These observations are also valid if we think in terms of essential nodes and essential branches. Thus if we let n_e represent the number of essential nodes and b_e the number of essential branches *where the current is unknown*, then we can apply Kirchhoff's current law at $n_e - 1$ nodes and Kirchhoff's voltage law around $b_e - (n_e - 1)$ loops or meshes.

A circuit may consist of disconnected parts so we must alert you to the fact that the statements pertaining to the number of equations that can be derived from Kirchhoff's current law $(n-1)$ and voltage law $[b-(n-1)]$ apply to connected circuits. If a circuit has n nodes and b branches and is made up of s parts, then the current law can be applied $n-s$ times and the voltage law $b-n+s$ times. We also make the observation that any two separate parts can be connected by a *single* conductor. This connection will always cause two nodes to form into one node. Furthermore, no current will exist in the single conductor. This means that any circuit made up of s disconnected parts can always be reduced to a connected circuit.

The Systematic Approach—An Illustration

Let us illustrate this systematic approach to deriving the simultaneous equations that describe a connected circuit in terms of its unknown currents. We will use the circuit shown in Fig. 4.3 and write the equations on the basis of essential nodes and branches. We have already noted that the circuit has four essential nodes and six essential branches where the current is unknown. The circuit in Fig. 4.3 has been redrawn in Fig. 4.4 with the six unknown currents defined by i_1 through i_6.

Three of the six simultaneous equations needed to describe this circuit are derived by applying Kirchhoff's current law to any three of the four essential nodes. We will use the nodes b, c, and e to get

$$\begin{aligned} -i_1 + i_2 + i_6 - I &= 0, \\ i_1 - i_3 - i_5 &= 0, \\ i_3 + i_4 - i_2 &= 0. \end{aligned} \tag{4.1}$$

The remaining three equations we derived by applying Kirchhoff's voltage law around three meshes. We have already noted that the circuit has four meshes. We will dismiss the mesh

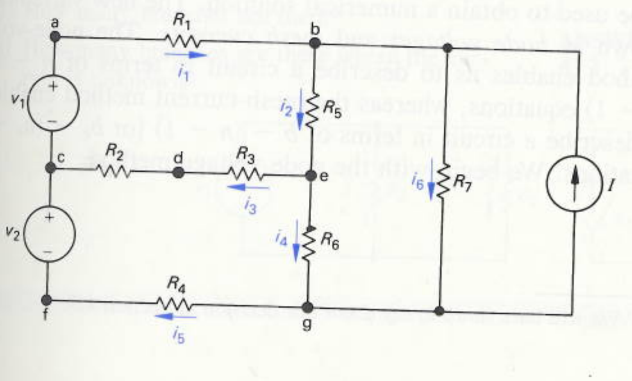


Figure 4.4 The circuit of Fig. 4.3 with six unknown branch currents defined.

$R_7 - I$ because we don't know the voltage across I.† Using the other three meshes in the order in which they are described in item (3) above gives us

$$\begin{aligned} R_1 i_1 + R_5 i_2 + i_3(R_2 + R_3) - v_1 &= 0, \\ -i_3(R_2 + R_3) + i_4 R_6 + i_5 R_4 - v_2 &= 0, \\ -i_2 R_5 + i_6 R_7 - i_4 R_6 &= 0. \end{aligned} \tag{4.2}$$

When the six simultaneous equations given by Eqs. (4.1) and (4.2) are rearranged to facilitate their solution, we get the set

$$\begin{aligned} -i_1 + i_2 + 0i_3 + 0i_4 + 0i_5 + i_6 &= I, \\ i_1 + 0i_2 - i_3 + 0i_4 - i_5 + 0i_6 &= 0, \\ 0i_1 - i_2 + i_3 + i_4 + 0i_5 + 0i_6 &= 0, \\ R_1 i_1 + R_5 i_2 + (R_2 + R_3)i_3 + 0i_4 + 0i_5 + 0i_6 &= v_1, \\ 0i_1 + 0i_2 - (R_2 + R_3)i_3 + R_6 i_4 + R_4 i_5 + 0i_6 &= v_2, \\ 0i_1 + R_5 i_2 + 0i_3 - R_6 i_4 + 0i_5 + R_7 i_6 &= 0. \end{aligned} \tag{4.3}$$

We note in passing that if we sum the current at the nth node (g in this example) we would get the equation

$$i_5 - i_4 - i_6 + I = 0. \tag{4.4}$$

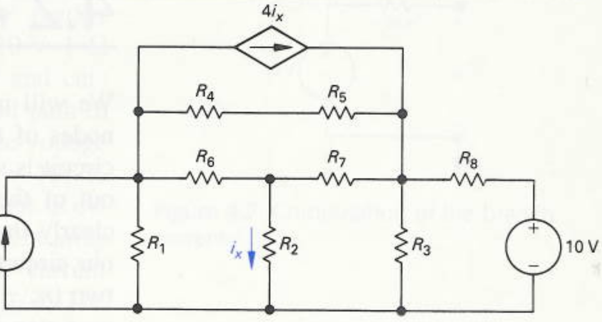
Equation (4.4) is not independent because it can be derived by adding the three equations in Eq. (4.1) and then multiplying the sum by -1. Thus Eq. (4.4) is a linear combination of the equations in Eq. (4.1) and therefore is not independent of them.

Now that we have illustrated a method for deriving the simultaneous equations that describe a circuit in terms of the unknown branch currents, we hasten to point out that we are not content to stop with this systematic formulation of the equations. We wish to carry our procedure one step further. We find that by introducing new variables we will be able to describe a circuit with just $n - 1$ equations or just $b - (n - 1)$ equations. Therefore these new variables allow us to obtain a solution by manipulating fewer equations, a desirable goal even if a computer is going to be used to obtain a numerical solution. The new variables are known as *node voltages* and *mesh currents*. The node-voltage method enables us to describe a circuit in terms of $n - 1$ (or $n_e - 1$) equations, whereas the mesh-current method enables us to describe a circuit in terms of $b - (n - 1)$ [or $b_e - (n_e - 1)$] equations. We begin with the node-voltage method.

† We will have more to say about this decision in Section 4.7.

DRILL EXERCISES

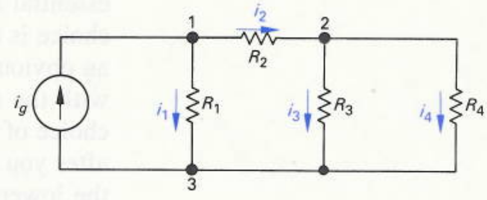
4.1 For the circuit shown, state the numerical value of the number of (a) branches, (b) branches where the current is unknown, (c) essential branches, (d) essential branches where the current is unknown, (e) nodes, (f) essential nodes, and (g) meshes.



ANSWER: (a) 11; (b) 9; (c) 9; (d) 7; (e) 6; (f) 4; (g) 6.

4.2 A current leaving a node is defined as positive.

a) Sum the currents at each node in the circuit shown.
b) Show that any one of the equations in part (a) can be derived from the remaining two equations.



ANSWER: (a) 1: $i_1 - i_g + i_2 = 0$; 2: $i_3 + i_4 - i_2 = 0$; 3: $i_g - i_1 - i_3 - i_4 = 0$. (b) To derive any one equation from the other two equations, simply add the two equations and then multiply the resulting sum by -1.

4.3 a) If only the essential nodes and branches are identified in the circuit of Drill Exercise 4.1, how many simultaneous equations are needed to describe the circuit?

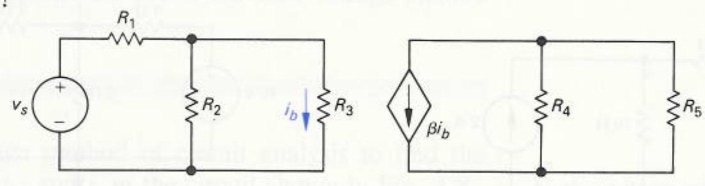
b) How many of these equations can be derived using Kirchhoff's current law?
c) How many need to be derived using Kirchhoff's voltage law?
d) What two meshes should be avoided in applying the voltage law?

ANSWER: (a) 7; (b) 3; (c) 4; (d) R_4–R_5–$4i_x$ and 8 A–R_1.

4.4 a) How many separate parts does the circuit shown have?

b) How many nodes?
c) How many independent current equations can be written?
d) How many branches are there?
e) How many branches are there where the current is unknown?
f) How many equations need to be written using the voltage law?
g) Assume that the lower node in each part of the circuit is jointed by a single conductor. Repeat the above calculations.

ANSWER: (a) 2; (b) 5; (c) 3; (d) 7; (e) 6; (f) 3; (g) 1, 4, 3, 7, 6, 3.



4.2 INTRODUCTION TO THE NODE-VOLTAGE METHOD

We will introduce the node-voltage method using the essential nodes of the circuit. To facilitate our discussion, an illustrative circuit is shown in Fig. 4.5. Our first step is to make a neat layout of the circuit so that no branches cross over and to mark clearly the essential nodes on the circuit diagram. We note that our circuit has three essential nodes ($n_e = 3$); therefore we need two ($n_e - 1$) node-voltage equations to describe the circuit. The next step in our systematic approach is to select one of the three essential nodes as a reference node. Although theoretically the choice is arbitrary, from a practical point of view, there is often an obvious choice for the reference node. For example, the node with the most branches is usually a good choice. The optimum choice of the reference node (if one exists) will become apparent after you have gained some experience using this method. Since the lower node connects the most branches in the circuit shown in Fig. 4.5, we will use it as the reference node. Once the reference node has been chosen, it is flagged by attaching the symbol ↓, as illustrated in Fig. 4.6.

Node voltage defined

After the reference node has been selected, the node voltages are defined on the circuit diagram. *A node voltage is defined as the voltage rise from the reference node to a nonreference node.* For the circuit under discussion, we must define two node voltages. These have been denoted as v_1 and v_2 in Fig. 4.6.

SPICE Using SPICE to describe a circuit: Sec. 2 (manual p. 3)

We are now ready to write the node-voltage equations. A node-voltage equation is generated by writing the current leaving each branch connected to a nonreference node as a function of the node voltages and then summing these currents to zero in accordance with Kirchhoff's current law. Let us demonstrate this using the circuit shown in Fig. 4.6. The current away from node 1 through the 1-Ω resistor will be the voltage drop across the resistor divided by the resistance (Ohm's law). The voltage drop across the resistor, in the direction of the current away

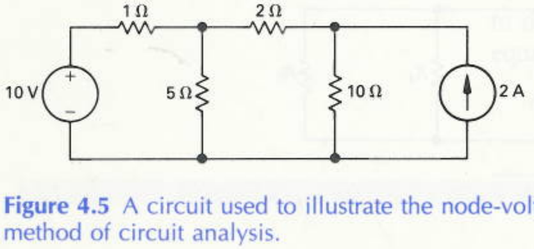


Figure 4.5 A circuit used to illustrate the node-voltage method of circuit analysis.

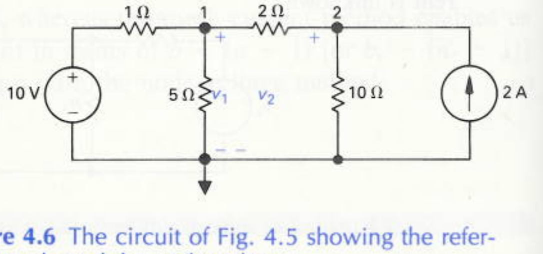


Figure 4.6 The circuit of Fig. 4.5 showing the reference node and the node voltages.

from the node, will be $v_1 - 10$. Therefore the current in the 1-Ω resistor is $(v_1 - 10)/1$. These observations are readily confirmed after referring to Fig. 4.7, in which the 10-V–1-Ω branch has been drawn with the appropriate voltages and current. Note that summing the voltages around the closed path in accordance with Kirchhoff's voltage law verifies that the voltage drop across the 1-Ω resistor in the direction of i is $(v_1 - 10)$ V.

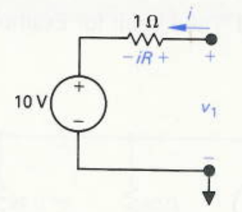


Figure 4.7 Computation of the branch current i.

This same thought process is used to obtain the current in every branch where the current is unknown. Thus the current away from node 1 through the 5-Ω resistor is $v_1/5$ and the current away from node 1 through the 2-Ω resistor is $(v_1 - v_2)/2$. The sum of the three currents leaving node 1 must equal zero; therefore the node-voltage equation derived at node 1 is

$$\frac{v_1 - 10}{1} + \frac{v_1}{5} + \frac{v_1 - v_2}{2} = 0. \tag{4.5}$$

The node-voltage equation written at node 2 is

$$\frac{v_2 - v_1}{2} + \frac{v_2}{10} - 2 = 0. \tag{4.6}$$

In studying Eq. (4.6), note that the first term is the current away from node 2 through the 2-Ω resistor, the second term is the current away from node 2 through the 10-Ω resistor, and the third term is the current away from node 2 through the current source.

Equations (4.5) and (4.6) are the two simultaneous equations that describe the circuit of Fig. 4.6 in terms of the node voltages v_1 and v_2. Solving for v_1 and v_2 yields

$$v_1 = \frac{100}{11} = 9.09 \text{ V}$$

and

$$v_2 = \frac{120}{11} = 10.91 \text{ V}.$$

It is important to note that once the node voltages are known, all the branch currents can be calculated. Once the branch currents are known, the branch voltages and powers can be calculated. Example 4.1 illustrates the use of the node-voltage method to analyze a circuit.

EXAMPLE 4.1

a) Use the node-voltage method of circuit analysis to find the branch currents i_a, i_b, and i_c in the circuit shown in Fig. 4.8.

b) Find the power associated with each source, and state whether the source is delivering or absorbing power.

Node-voltage method

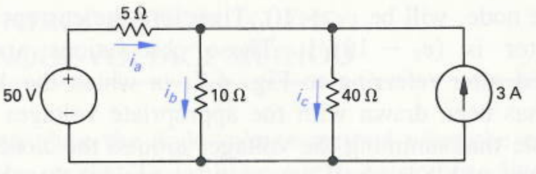


Figure 4.8 The circuit for Example 4.1.

SOLUTION

a) We begin by noting that the circuit has two essential nodes; thus we need to write a single node-voltage expression. We will select the lower node as the reference node and define our unknown node voltage as v_1. These decisions are illustrated in Fig. 4.9. Summing the currents away from node 1 generates the following node-voltage equation:

$$\frac{v_1 - 50}{5} + \frac{v_1}{10} + \frac{v_1}{40} - 3 = 0.$$

Solving for v_1 we obtain

$$v_1 = 40 \text{ V}.$$

It follows directly that

$$i_a = \frac{50 - 40}{5} = 2 \text{ A},$$

$$i_b = \frac{40}{10} = 4 \text{ A},$$

$$i_c = \frac{40}{40} = 1 \text{ A}.$$

b) The power associated with the 50-V source is

$$p_{50\,\text{V}} = -50i_a = -100 \text{ W (delivering)}.$$

The power associated with the 3-A source is

$$p_{3\,\text{A}} = -3v_1 = -3(40) = -120 \text{ W (delivering)}.$$

As a check for our calculations, we note that the total delivered power is 220 W. The total power absorbed by the three resistors is 4(5) + 16(10) + 1(40), which is 220 W, as it must be.

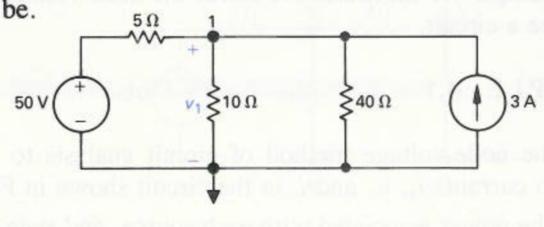


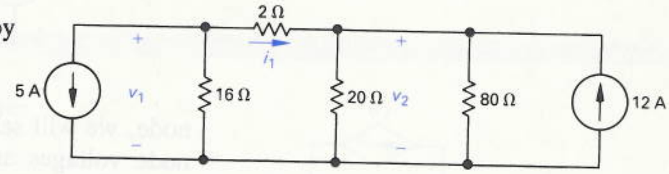
Figure 4.9 The circuit of Fig. 4.8 showing the reference node and the unknown node voltage v_1.

DRILL EXERCISES

4.5 a) For the circuit shown, use the node-voltage method to find v_1, v_2, and i_1.

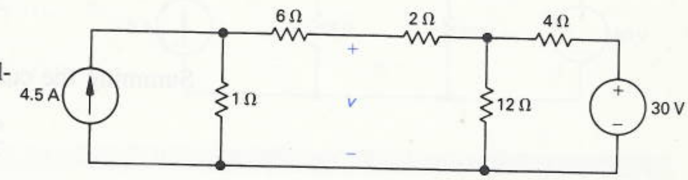
b) How much power is delivered to the circuit by the 12-A source?

c) Repeat for the 5-A source.



ANSWER: (a) 48 V, 64 V, −8 A; (b) 768 W; (c) −240 W.

4.6 Use the node-voltage method to find v in the following circuit.



ANSWER: 15 V.

4.3 THE NODE-VOLTAGE METHOD AND DEPENDENT SOURCES

If the circuit contains dependent sources, the node-voltage equations must be supplemented with the constraint equations imposed by the presence of the dependent sources. Example 4.2 illustrates the application of the node-voltage method to a circuit containing a dependent source.

EXAMPLE 4.2

Use the node-voltage method to find the power dissipated in the 5-Ω resistor in the circuit shown in Fig. 4.10.

Node-voltage method: dependent source

SOLUTION

We begin by noting that our circuit has three essential nodes and, therefore, we will need two node-voltage equations to describe the circuit. Since four branches terminate on the lower

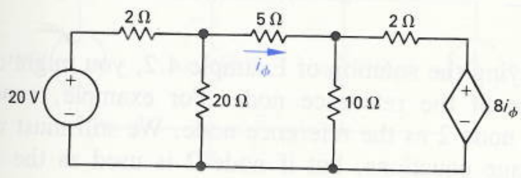


Figure 4.10 The circuit for Example 4.2.

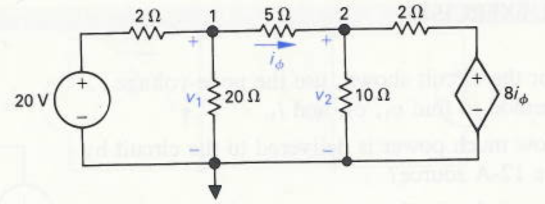


Figure 4.11 The circuit of Fig. 4.10 with reference node and node voltages.

node, we will select it as our reference node. The two unknown node voltages are defined on the circuit shown in Fig. 4.11. Summing the currents away from node 1 generates the equation

$$\frac{v_1 - 20}{2} + \frac{v_1}{20} + \frac{v_1 - v_2}{5} = 0.$$

Summing the currents away from node 2 yields

$$\frac{v_2 - v_1}{5} + \frac{v_2}{10} + \frac{v_2 - 8i_\phi}{2} = 0.$$

As written, our two node-voltage equations contain three unknowns, namely, v_1, v_2, and i_ϕ. To eliminate i_ϕ we must express this controlling current in terms of the node voltages. Thus we have

$$i_\phi = \frac{v_1 - v_2}{5}.$$

When this relationship is substituted into the node 2 equation, the two node-voltage equations simplify to

$$0.75v_1 - 0.2v_2 = 10,$$

$$-v_1 + 1.6v_2 = 0.$$

Solving for v_1 and v_2 gives

$$v_1 = 16 \text{ V} \quad \text{and} \quad v_2 = 10 \text{ V}.$$

It follows directly that

$$i_\phi = \frac{16 - 10}{5} = 1.2 \text{ A}$$

and

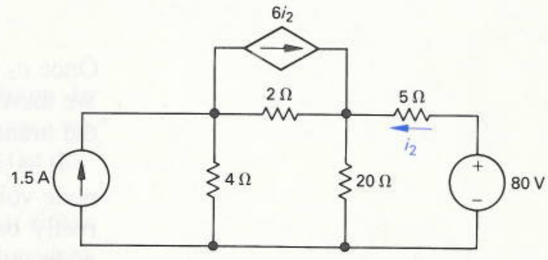
$$p_{5\Omega} = (1.44)(5) = 7.2 \text{ W}.$$

In studying the solution of Example 4.2, you might challenge the choice of the reference node. For example, consider the choice of node 2 as the reference node. We still must write two node-voltage equations, but if node 2 is used as the reference node, we need only solve for one of the unknown node voltages —specifically, the node voltage across the 5-Ω resistor. (See Problem 4.3.)

DRILL EXERCISES

4.7 a) Use the node-voltage method to find the power associated with each source in the following circuit.

b) State whether the source is delivering power to the circuit or extracting power from the circuit.



ANSWER: (a) $p_{1.5\,\text{A}} = 15$ W; $p_{6i_2} = 1200$ W; $p_{80\text{V}} = 320$ W. (b) All sources are delivering power to the circuit.

4.4 THE NODE-VOLTAGE METHOD: SOME SPECIAL CASES

When a voltage source is the only element between two essential nodes, the node-voltage method requires some additional manipulations. The nature of the problem can be seen from the circuit shown in Fig. 4.12. The reference node and the node voltages are as shown in the figure. We can see that we have a problem in writing the expression for the current leaving node 1 through the independent voltage source. The problem arises because there is no resistance in series with the 100-V source. At first glance, it may appear that the current in this branch is infinite, as implied by the expression $(v_1 - 100)/0$. However, closer inspection shows that v_1 must be 100 V and therefore we really have the indeterminate form 0/0. It is the observation that v_1 equals 100 V that allows us to apply the node-voltage method with no further difficulty. That is, once we recognize that v_1 equals 100 V, we see that we have only one unknown node voltage (v_2) and that

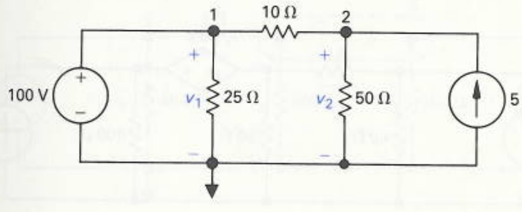


Figure 4.12 A circuit with a known node voltage.

therefore we can solve this particular circuit by solving a single node-voltage equation. At node 2 we have

$$\frac{v_2 - v_1}{10} + \frac{v_2}{50} - 5 = 0. \tag{4.7}$$

But $v_1 = 100$ V; therefore Eq. (4.7) can be solved for v_2:

$$v_2 = 125 \text{ V}. \tag{4.8}$$

Once v_2 is known, we can calculate the current in every branch. We leave it to the reader to verify that the current into node 1 in the branch containing the independent voltage source is 1.5 A.

It is important to note at this point that, when we use the node-voltage method, any voltage sources that are connected directly between essential nodes reduce the number of unknown node voltages. This follows because whenever a voltage source connects two essential nodes, it constrains the difference between the node voltages at the essential nodes to equal the voltage of the source.

A numerical example

Assume that the circuit shown in Fig. 4.13 is to be analyzed using the node-voltage method. In studying the circuit, we note that there are four essential nodes; thus we anticipate writing three node-voltage equations. However, further study of the circuit reveals that two essential nodes are connected by an independent voltage source and two other essential nodes are tied together via a current-controlled dependent voltage source. Therefore we can deduce that there is really only one unknown node voltage. For example, note that if the voltage across the 50-Ω resistor is known, the voltage across the 100-Ω resistor is also known because of the presence of the dependent voltage source. When we are choosing which node to use as the reference node, several thoughts come to mind. Either node on each side of the dependent voltage source looks attractive because, if chosen, one of the node voltages would be known to be either $+10i_\phi$ (left node is the reference) or $-10i_\phi$ (right node is the reference). The lower node looks even more attractive because, if chosen, one node voltage is immediately known (50 V) and five branches terminate there. We therefore opt for the lower node as the reference.

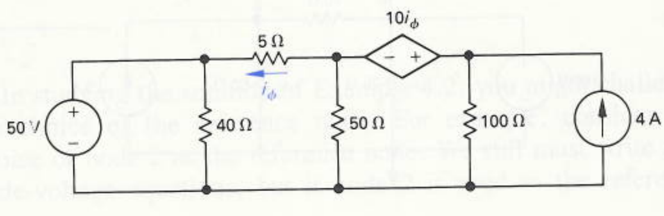


Figure 4.13 A circuit with a dependent voltage source connected between nodes.

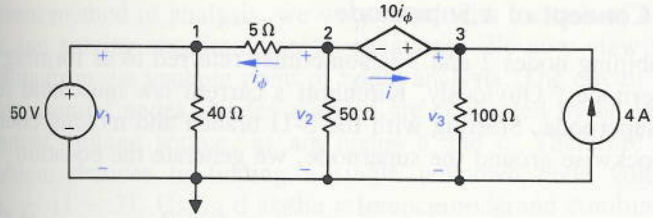


Figure 4.14 The circuit of Fig. 4.13 defining the selected node voltages.

The circuit has been redrawn in Fig. 4.14. In addition to flagging the reference node and defining the node voltages, we have introduced the current i, which is needed to support the discussion that follows.

In writing the appropriate node-voltage equation at either node 2 or 3 we cannot express the current in the dependent voltage source branch as a function of the node voltages v_2 and v_3. To solve our dilemma, we introduce the unknown current i and then promptly eliminate it from our equations. Thus at node 2 we have

$$\frac{v_2 - v_1}{5} + \frac{v_2}{50} + i = 0, \tag{4.9}$$

and at node 3,

$$\frac{v_3}{100} - i - 4 = 0. \tag{4.10}$$

We eliminate i by simply adding Eqs. (4.9) and (4.10) to get

$$\frac{v_2 - v_1}{5} + \frac{v_2}{50} + \frac{v_3}{100} - 4 = 0. \tag{4.11}$$

We note in passing that Eq. (4.11) can be written directly, without resorting to the intermediate step represented by Eqs. (4.9) and (4.10). To write Eq. (4.11) directly, we visualize nodes 2 and 3 as composing a single node and simply sum the currents away from the node in terms of the node voltages v_2 and v_3. This point of view is shown in Fig. 4.15.

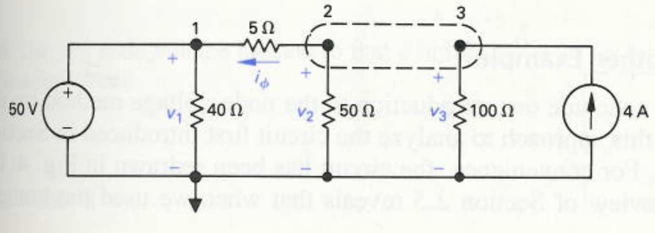


Figure 4.15 Visualizing nodes 2 and 3 as a "supernode."

The Concept of a Supernode

Supernode

Combining nodes 2 and 3 is sometimes referred to as forming a "supernode." Obviously, Kirchhoff's current law must hold for the supernode. Starting with the 5-Ω branch and moving counterclockwise around the supernode, we generate the equation

$$\frac{v_2 - v_1}{5} + \frac{v_2}{50} + \frac{v_3}{100} - 4 = 0, \quad \textbf{(4.12)}$$

which is identical to Eq. (4.11). Therefore the motivation for creating the supernode is apparent. The supernode concept can be used whenever two essential nodes are connected by a voltage source element.

Once Eq. (4.11) (or Eq. 4.12) has been derived, the next step is to reduce the expression to a single unknown node voltage. First we eliminate v_1 from the equation since v_1 is known to be 50 V. Next we can express v_3 as a function of v_2:

$$v_3 = v_2 + 10i_\phi. \quad \textbf{(4.13)}$$

The current controlling the dependent voltage source is now expressed as a function of the node voltages, as follows:

$$i_\phi = \frac{v_2 - 50}{5}. \quad \textbf{(4.14)}$$

Using Eqs. (4.13) and (4.14) along with the fact that v_1 equals 50 V reduces Eq. (4.11) to

$$v_2\left(\frac{1}{50} + \frac{1}{5} + \frac{1}{100} + \frac{10}{500}\right) = 10 + 4 + 1$$

$$v_2(0.25) = 15$$

$$v_2 = 60 \text{ V}.$$

It follows directly from Eqs. (4.13) and (4.14) that

$$i_\phi = \frac{60 - 50}{5} = 2 \text{ A}$$

and

$$v_3 = 60 + 20 = 80 \text{ V}.$$

Another Example

We conclude our introduction to the node-voltage method by using this approach to analyze the circuit first introduced in Section 2.5. For convenience, the circuit has been redrawn in Fig. 4.16. A review of Section 2.5 reveals that when we used the branch-current method of analysis, we were faced with the task of writing and solving six simultaneous equations. We now view this circuit from the vantage point of nodal analysis. The circuit has four essential nodes. Nodes a and d are connected by an independent voltage source, as are nodes b and c. Therefore the problem reduces to finding a single unknown node voltage $[(n_e - 1) - 2]$. Using d as the reference node and combining nodes b and c into a supernode, we have

$$\frac{v_b}{R_2} + \frac{v_b - V_{CC}}{R_1} + \frac{v_c}{R_E} - \beta i_B = 0. \quad (4.15)$$

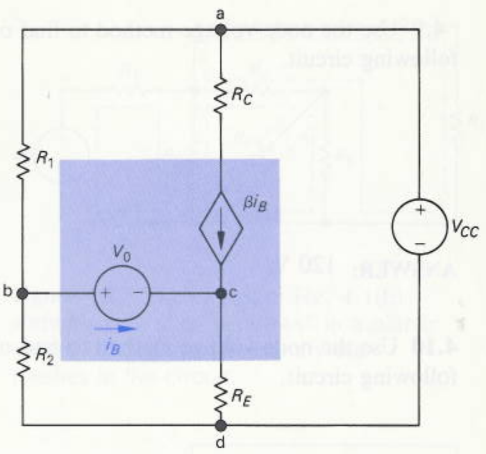


Figure 4.16 The transistor amplifier circuit of Fig. 2.18.

In writing Eq. (4.15), we have used v_b and v_c to denote the voltage rise from the reference node to nodes b and c, respectively. We have also used the fact that the voltage rise from the reference node to node a is V_{CC}. Now we can eliminate both v_c and i_B from Eq. (4.15) by noting that

$$v_c = (i_B + \beta i_B)R_E \quad (4.16)$$

and

$$v_c = v_b - V_0. \quad (4.17)$$

When Eqs. (4.16) and (4.17) are substituted into Eq. (4.15), we get

$$v_b\left[\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{(1+\beta)R_E}\right] = \frac{V_{CC}}{R_1} + \frac{V_0}{(1+\beta)R_E}. \quad (4.18)$$

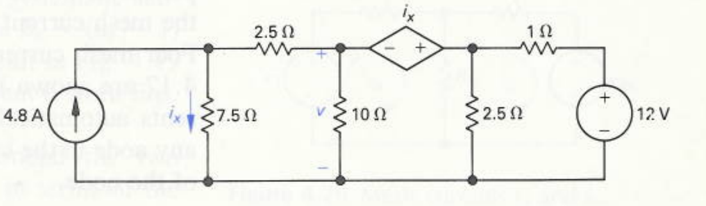
Solving Eq. (4.18) for v_b yields

$$v_b = \frac{V_{CC}R_2(1+\beta)R_E + V_0R_1R_2}{R_1R_2 + (1+\beta)R_E(R_1+R_2)}. \quad (4.19)$$

Using the node-voltage method to analyze the circuit in Fig. 4.15 has reduced our problem from manipulating six simultaneous equations to manipulating three simultaneous equations. (See Problem 2.11.) We leave it to the reader to verify that when Eq. (4.19) is combined with Eqs. (4.16) and (4.17) the solution for i_B is identical to Eq. (2.27). (See Problem 4.15.)

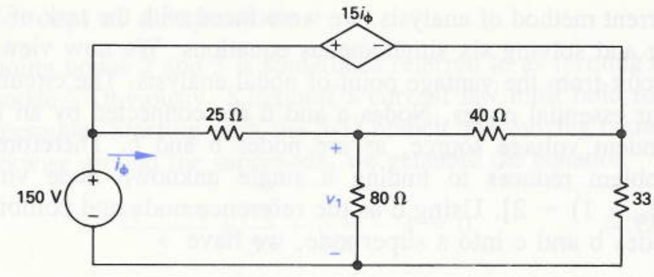
DRILL EXERCISES

4.8 Use the node-voltage method to find v in the following circuit.



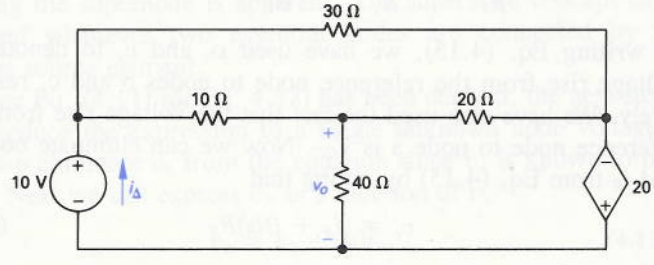
ANSWER: 8 V.

4.9 Use the node-voltage method to find v_1 in the following circuit.



ANSWER: 120 V.

4.10 Use the node-voltage method to find v_o in the following circuit.



ANSWER: 24 V.

4.5 INTRODUCTION TO MESH CURRENTS

Meshes and windows

As stated in Section 4.1, the mesh-current method of circuit analysis enables us to describe a circuit in terms of $b - (n - 1)$ or $b_e - (n_e - 1)$ equations. For planar networks, the meshes in the network are identical to the "windows" that are formed when the network is drawn with no branches crossing over each other. The circuit in Fig. 4.1(b) is shown redrawn in Fig. 4.17, where the "windows" are identified by the closed dashed paths. Note in Fig. 4.17 that there are seven essential branches where the current is unknown. Since the circuit contains four essential nodes, we need to write four $[7 - (4 - 1)]$ mesh-current equations.

Mesh current defined

A *mesh current* is defined as the current that exists only in the perimeter of a mesh. It is indicated on a circuit diagram as either a closed solid line or an almost-closed solid line that follows the perimeter of the appropriate mesh. The reference direction for the mesh current is indicated by an arrowhead on the solid line. Four mesh currents that are used to describe the circuit in Fig. 4.17 are shown in Fig. 4.18. Note that by definition mesh currents automatically satisfy Kirchhoff's current law. That is, at any node in the circuit, a given mesh current is both into and out of the node.

In studying Fig. 4.18, note that from the definition of a mesh current it is not always possible to identify it in terms of a branch current. For example, in Fig. 4.18 the mesh current i_2 is not equal to any branch current, whereas mesh currents i_1, i_3, and i_4 can be identified with branch currents. Thus it is not always possible to measure a mesh current. For example, in Fig. 4.18 there is no place where an ammeter can be inserted into the circuit to measure the mesh current i_2. The fact that a mesh current can be a fictitious quantity does not mean that it is a useless concept. On the contrary, it is very useful to us in circuit analysis.

The mesh-current method of circuit analysis evolves quite naturally from the branch-current equations. The circuit in Fig. 4.19 can be used to show the evolution of the mesh-current technique. We begin by using the branch currents (i_1, i_2, and i_3) to formulate the set of independent equations. For this circuit, $b_e = 3$ and $n_e = 2$. We can write only one independent current equation; therefore we will need two independent voltage equations. Applying Kirchhoff's current law to the upper node and Kirchhoff's voltage law around the two meshes generates the following set of equations:

$$i_1 = i_2 + i_3, \tag{4.20}$$

$$v_1 = i_1 R_1 + i_3 R_3, \tag{4.21}$$

$$-v_2 = i_2 R_2 - i_3 R_3. \tag{4.22}$$

We can reduce this set of three equations to a set of two equations by solving Eq. (4.20) for i_3 and then substituting this expression for i_3 into Eqs. (4.21) and (4.22). The result is

$$v_1 = i_1(R_1 + R_3) - i_2 R_3, \tag{4.23}$$

$$-v_2 = i_1 R_3 + i_2(R_2 + R_3). \tag{4.24}$$

We can solve Eqs. (4.23) and (4.24) for i_1 and i_2 to reduce the original problem of solving three simultaneous equations to a problem of solving two simultaneous equations. We have derived Eqs. (4.23) and (4.24) by substituting the $n_e - 1$ current equations into the $b_e - (n_e - 1)$ voltage equations. The value of the mesh-current method lies in the fact that by defining mesh currents we *automatically* eliminate the $n_e - 1$ current equations. Thus the mesh-current method is equvialent to a systematic substitution of the $n_e - 1$ current equations into the $b_e - (n_e - 1)$ voltage equations. The mesh currents for the circuit in Fig. 4.19 that are equivalent to eliminating the branch current i_3 from Eqs. (4.21) and (4.22) are shown in Fig. 4.20.

Now we apply Kirchhoff's voltage law around the two meshes, expressing all voltages across resistors in terms of the

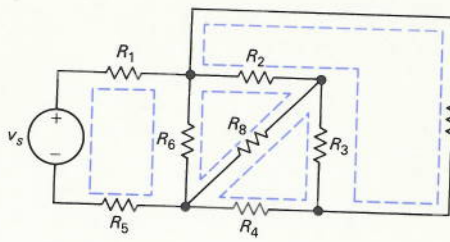


Figure 4.17 The circuit of Fig. 4.1(b) showing how the "windows" in a planar circuit can be used to identify the meshes in the circuit.

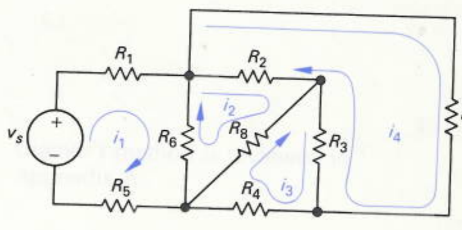


Figure 4.18 The circuit of Fig. 4.17 with the mesh currents defined.

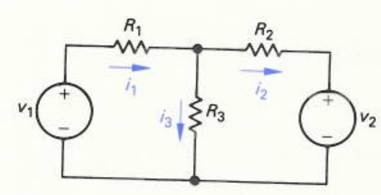


Figure 4.19 A circuit used to illustrate the development of the mesh-current method of circuit analysis.

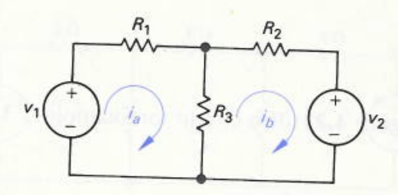


Figure 4.20 Mesh currents i_a and i_b.

mesh currents, to get the equations

$$v_1 = i_a R_1 + (i_a - i_b)R_3 \quad (4.25)$$

and

$$-v_2 = (i_b - i_a)R_3 + i_b R_2. \quad (4.26)$$

Collecting the coefficients of i_a and i_b in Eqs. (4.25) and (4.26) gives us

$$v_1 = i_a(R_1 + R_3) - i_b R_3 \quad (4.27)$$

and

$$-v_2 = -i_a R_3 + i_b(R_2 + R_3). \quad (4.28)$$

When we compare Eqs. (4.27) and (4.28) with Eqs. (4.23) and (4.24), we see that they are identical in form, with the mesh currents i_a and i_b replacing the branch currents i_1 and i_2. By comparing the circuits in Figs. 4.19 and 4.20, we can also see that the branch currents can be expressed in terms of the mesh currents by inspection; hence

$$i_1 = i_a, \quad (4.29)$$

$$i_2 = i_b, \quad (4.30)$$

$$i_3 = i_a - i_b. \quad (4.31)$$

The ability to write Eqs. (4.29)–(4.31) by inspection is crucial to the mesh-current method of circuit analysis. Once we know the mesh currents, we also know the branch currents. And once we know the branch currents, we can compute any voltages or powers of interest.

One final comment about the mesh-current method before we illustrate this approach with a numerical example. Since the meshes have been defined as the "windows" of a planar circuit, we guarantee that the set of mesh-current equations that describe the circuit will be an independent set. We shall prove that the mesh-current equations form an independent set in Chapter 5.

EXAMPLE 4.3

Mesh current method

a) Use the mesh-current method to determine the power associated with each voltage source in the circuit shown in Fig. 4.21.

b) Calculate the voltage v_0 across the 8-Ω resistor.

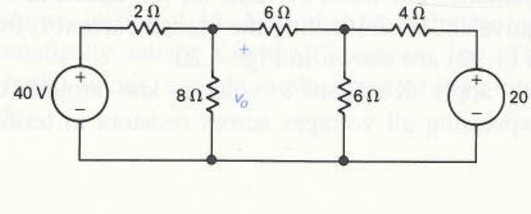


Figure 4.21 The circuit for Example 4.3.

SOLUTION

a) To calculate the power associated with each source, we need to know the current in each source. In studying the circuit, we see that these source currents will be identical to mesh currents. We also note that our circuit has seven branches where the current is unknown and five nodes. Therefore we need three mesh-current equations to describe the circuit, that is, $b - (n - 1) = 7 - (5 - 1) = 3$. The three mesh currents that we are using to describe the circuit in Fig. 4.21 are shown in Fig. 4.22.

If we assume that the voltage drops are positive, the three mesh equations are

$$\begin{aligned} -40 + 2i_a + 8(i_a - i_b) &= 0, \\ 8(i_b - i_a) + 6i_b + 6(i_b - i_c) &= 0, \\ 6(i_c - i_b) + 4i_c + 20 &= 0. \end{aligned} \quad (4.32)$$

Equations (4.32) can now be reorganized in anticipation of using Cramer's method for solving simultaneous equations. We get

Cramer's method is reviewed in Appendix A

$$\begin{aligned} 10i_a - 8i_b + 0i_c &= 40, \\ -8i_a + 20i_b - 6i_c &= 0, \\ 0i_a - 6i_b + 10i_c &= -20. \end{aligned} \quad (4.33)$$

The characteristic determinant is

$$\Delta = \begin{vmatrix} 10 & -8 & 0 \\ -8 & 20 & -6 \\ 0 & -6 & 10 \end{vmatrix}$$

$$= 10(200 - 36) + 8(-80)$$

$$= 1640 - 640 = 1000.$$

The three mesh currents are

$$i_a = \frac{\begin{vmatrix} 40 & -8 & 0 \\ 0 & 20 & -6 \\ -20 & -6 & 10 \end{vmatrix}}{1000}$$

$$= \frac{40(200 - 36) - 20(48)}{1000}$$

$$= 5.6 \text{ A};$$

$$i_b = \frac{\begin{vmatrix} 10 & 40 & 0 \\ -8 & 0 & -6 \\ 0 & -20 & 10 \end{vmatrix}}{1000}$$

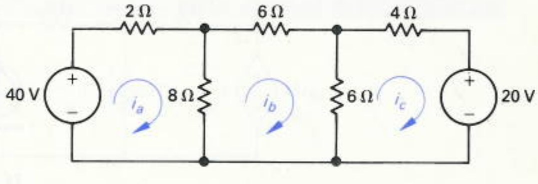


Figure 4.22 Three mesh currents used to analyze the circuit of Fig. 4.21.

$$= \frac{10(-120) + 8(400)}{1000}$$
$$= 2.0 \text{ A};$$

$$i_c = \frac{\begin{vmatrix} 10 & -8 & 40 \\ -8 & 20 & 0 \\ 0 & -6 & -20 \end{vmatrix}}{1000}$$
$$= \frac{10(-400) + 8(160 + 240)}{1000}$$
$$= -0.80 \text{ A}.$$

Now since the mesh current i_a is identical with the branch current in the 40-V source, the power associated with this source is

$$p_{40\text{V}} = -40i_a = -224 \text{ W}.$$

The minus sign tells us that this source is delivering power to the network. The current in the 20-V source is identical to the mesh current i_c; therefore

$$p_{20\text{V}} = 20i_c = -16 \text{ W}.$$

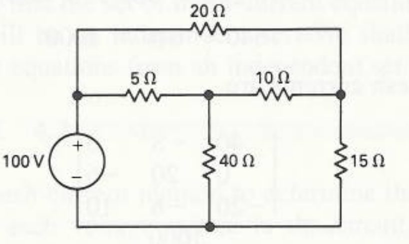
The 20-V source is also delivering power to the network.

b) The branch current in the 8-Ω resistor in the direction of the voltage drop v_0 is $i_a - i_b$. Therefore

$$v_0 = 8(i_a - i_b) = 8(3.6) = 28.8 \text{ V}.$$

DRILL EXERCISES

4.11 Use the mesh-current method to find (a) the power delivered to the circuit by the 100-V source and (b) the power dissipated in the 15-Ω resistor.



ANSWER: (a) 600 W; (b) 240 W.

4.6 THE MESH-CURRENT METHOD AND DEPENDENT SOURCES

If the circuit contains dependent sources, the mesh-current equations must be supplemented by the appropriate constraint equations imposed by the presence of the dependent source or sources. Example 4.4 illustrates the application of the mesh-current method when the circuit includes a dependent source.

EXAMPLE 4.4

Mesh current method: dependent source

Use the mesh-current method of circuit analysis to determine the power dissipated in the 4-Ω resistor in the circuit shown in Fig. 4.23.

SOLUTION

Our circuit has six branches where the current is unknown and four nodes; therefore we know that we need three mesh currents to describe the circuit. They are defined on the circuit shown in Fig. 4.24. The three mesh-current equations are

$$\begin{aligned} 50 &= 5(i_1 - i_2) + 20(i_1 - i_3), \\ 0 &= 5(i_2 - i_1) + 1i_2 + 4(i_2 - i_3), \\ 0 &= 20(i_3 - i_1) + 4(i_3 - i_2) + 15i_\phi. \end{aligned} \tag{4.34}$$

The branch current controlling the dependent voltage source is now expressed in terms of the mesh currents as

$$i_\phi = i_1 - i_3, \tag{4.35}$$

which is the supplemental equation imposed by the presence of the dependent source. When Eq. (4.35) is substituted into Eqs. (4.34) and the coefficients of i_1, i_2, and i_3 are collected in each equation, we get

$$\begin{aligned} 50 &= 25i_1 - 5i_2 - 20i_3, \\ 0 &= -5i_1 + 10i_2 - 4i_3, \\ 0 &= -5i_1 - 4i_2 + 9i_3. \end{aligned}$$

The characteristic determinant is

$$\Delta = \begin{vmatrix} 25 & -5 & -20 \\ -5 & 10 & -4 \\ -5 & -4 & 9 \end{vmatrix}.$$

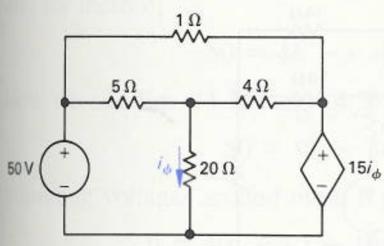


Figure 4.23 The circuit for Example 4.4.

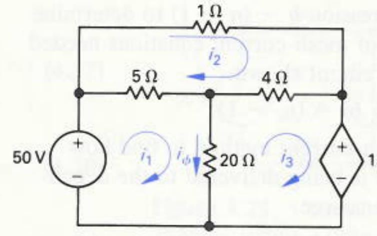


Figure 4.24 The circuit of Fig. 4.23 showing the three mesh currents.

Expanding the characteristic determinant by the first column gives

$$\Delta = 25(90 - 16) + 5(-45 - 80) - 5(20 + 200) = 125.$$

Since we are calculating the power dissipated in the 4-Ω resistor, we compute the mesh currents i_2 and i_3. We have

$$i_2 = \frac{\begin{vmatrix} 25 & 50 & -20 \\ -5 & 0 & -4 \\ -5 & 0 & 9 \end{vmatrix}}{125} = \frac{-50(-45-20)}{125} = 26 \text{ A},$$

and

$$i_3 = \frac{\begin{vmatrix} 25 & -5 & 50 \\ -5 & 10 & 0 \\ -5 & -4 & 0 \end{vmatrix}}{125} = \frac{50(20+50)}{125} = 28 \text{ A}.$$

The current in the 4-Ω resistor oriented from left to right is $i_3 - i_2$, or 2 A. Therefore the power dissipated is

$$p_{4\Omega} = (i_3 - i_2)^2(4) = (2)^2(4) = 16 \text{ W}.$$

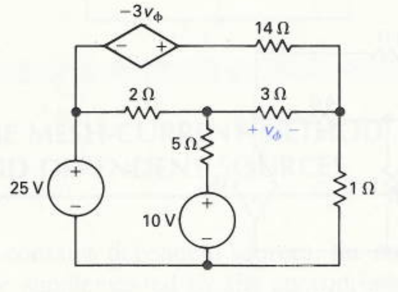
It is worth noting that if you were not told to use the mesh-current method, you could probably have chosen to use the node-voltage method since the presence of two voltage sources between essential nodes reduces the problem to finding one unknown node voltage. More about making choices later.

DRILL EXERCISES

4.12 a) Use the expression $b - (n - 1)$ to determine the number of mesh-current equations needed to solve the circuit shown.

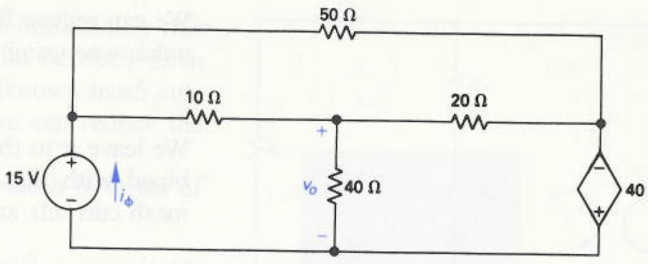
b) Repeat using $b_e - (n_e - 1)$.

c) Use the mesh-current method to find how much power is being delivered to the dependent voltage source.



ANSWER: (a) 3; (b) 3; (c) −36 W.

4.13 Use the mesh-current method to find v_o in the following circuit.



ANSWER: 20 V.

4.7 THE MESH-CURRENT METHOD: SOME SPECIAL CASES

When a branch includes a current source, the mesh-current method requires some additional manipulations. The nature of the problem can be understood from the circuit shown in Fig. 4.25. The mesh currents i_a, i_b, and i_c, as well as the voltage across the 5-A current source, have been defined in the figure to facilitate the discussion. In analyzing the circuit, we note that there are five essential branches where the current is unknown; furthermore, the circuit has four essential nodes. It follows that we need to write two [5 − (4 − 1)] mesh-current equations in order to solve the circuit. If we use the windows to define the meshes, we see that the three unknown mesh currents reduce to two unknown mesh currents because the current source coupling meshes a and c limits the differences between i_c and i_a to equal 5 A. However, when we attempt to sum the voltages around either mesh a or mesh c, we are forced to introduce the unknown voltage across the 5-A current source into our equations. We can eliminate this unknown voltage by simply introducing it into both mesh equations and then adding the two equations. Thus for mesh a we have

$$100 = 3(i_a - i_b) + v + 6i_a, \tag{4.36}$$

and for mesh c,

$$-50 = 4i_c - v + 2(i_c - i_b). \tag{4.37}$$

Now we add Eqs. (4.36) and (4.37) to obtain

$$50 = 9i_a - 5i_b + 6i_c. \tag{4.38}$$

Summing voltages around mesh b gives us

$$0 = 3(i_b - i_a) + 10i_b + 2(i_b - i_c). \tag{4.39}$$

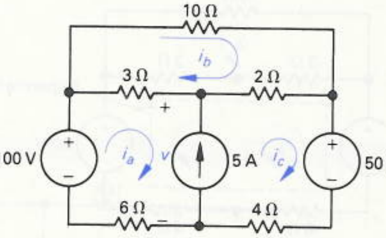


Figure 4.25 A circuit illustrating mesh analysis when a branch contains an independent current source.

We can reduce Eqs. (4.38) and (4.39) to two equations and two unknowns using the constraint that

$$i_c - i_a = 5. \tag{4.40}$$

We leave it to the reader to verify that when Eq. (4.40) is combined with Eqs. (4.38) and (4.39) the solutions for the three mesh currents are

$$i_a = 1.75 \text{ A},$$

$$i_b = 1.25 \text{ A},$$

and

$$i_c = 6.75 \text{ A}.$$

The Concept of a Supermesh

Supermesh

Equation (4.38) can be derived without introducing the unknown voltage v by using the concept of a *supermesh*. To create a supermesh, we mentally remove the current source from the circuit by simply avoiding this branch when writing the mesh-current equations. The voltages around the supermesh are expressed in terms of the mesh currents defined by the original windows of the circuit. The supermesh concept is illustrated in Fig. 4.26. When we sum the voltages around the supermesh denoted by the dashed line in Fig. 4.26, we obtain the equation

$$-100 + 3(i_a - i_b) + 2(i_c - i_b) + 50 + 4i_c + 6i_a = 0, \tag{4.41}$$

which reduces to

$$50 = 9i_a - 5i_b + 6i_c. \tag{4.42}$$

If we compare Eqs. (4.42) and (4.38), we see that they are identical. Thus the supermesh has eliminated the need for introducing the unknown voltage across the current source into our equations.

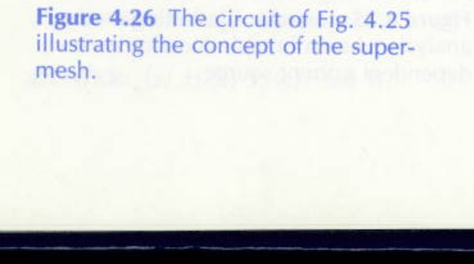


Figure 4.26 The circuit of Fig. 4.25 illustrating the concept of the supermesh.

Mesh-Current Analysis of a Familiar Circuit

We can use the circuit first introduced in Section 2.5 (Fig. 2.18) to illustrate the mesh-current method when a branch contains a dependent current source. The circuit is shown redrawn in Fig. 4.27 with the three mesh currents denoted as i_a, i_b, and i_c. When we study the circuit in Fig. 4.27 with the decision to use the mesh-current method in mind, we note that it has four essential nodes and five essential branches where the current is unknown. Therefore we know that the circuit can be analyzed in terms of two [5 − (4 − 1)] mesh-current equations. Although three mesh currents are defined in Fig. 4.27, we see immediately that the dependent current source forces a constraint between mesh currents i_a and i_c so that we have only two unknown mesh currents. Using the concept of the supermesh, we can redraw the circuit as shown in Fig. 4.28.

Now we sum the voltages around the supermesh in terms of the mesh currents i_a, i_b, and i_c. We get

$$R_1 i_a + V_{CC} + R_E(i_c - i_b) - V_0 = 0. \quad (4.43)$$

The mesh b equation is

$$R_2 i_b + V_0 + R_E(i_b - i_c) = 0. \quad (4.44)$$

The constraint imposed by the dependent current source is

$$\beta i_B = i_a - i_c. \quad (4.45)$$

The branch current controlling the dependent current source, expressed as a function of the mesh currents, is

$$i_B = i_b - i_a. \quad (4.46)$$

From Eqs. (4.45) and (4.46) we have

$$i_c = (1 + \beta)i_a - \beta i_b. \quad (4.47)$$

We can now use Eq. (4.47) to eliminate i_c from Eqs. (4.43) and (4.44). We get

$$[R_1 (1 + \beta)R_E]i_a - (1 + \beta)R_E i_b = V_0 - V_{CC}, \quad (4.48)$$

$$-(1 + \beta)R_E i_a + [R_2 + (1 + \beta)R_E]i_b = -V_0. \quad (4.49)$$

We will leave it to the reader to verify that the solution of Eqs. (4.48) and (4.49) for i_a and i_b gives

$$i_a = \frac{V_0 R_2 - V_{CC} R_2 - V_{CC}(1 + \beta)R_E}{R_1 R_2 + (1 + \beta)R_E(R_1 + R_2)}, \quad (4.50)$$

$$i_b = \frac{-V_0 R_1 - (1 + \beta)R_E V_{CC}}{R_1 R_2 + (1 + \beta)R_E(R_1 + R_2)}. \quad (4.51)$$

We also leave it to the reader to verify that when Eqs. (4.50) and (4.51) are used to find i_B, the result is the same as that given by Eq. (2.27).

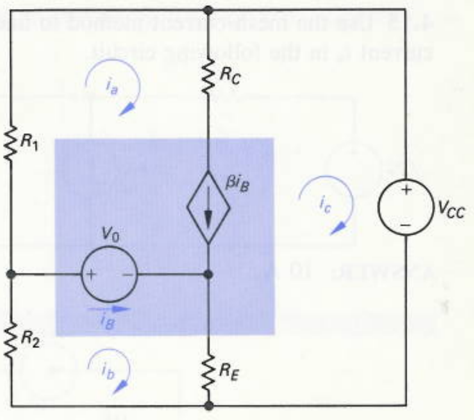


Figure 4.27 The circuit of Fig. 2.18 showing the mesh currents i_a, i_b, and i_c.

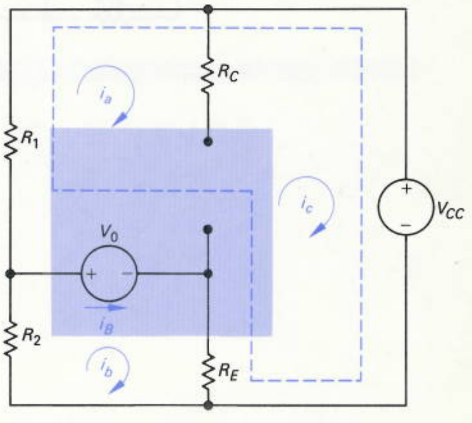
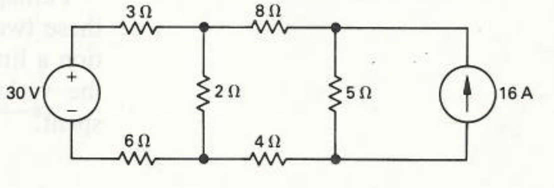


Figure 4.28 The circuit of Fig. 4.27 showing the supermesh created by the presence of the dependent current source.

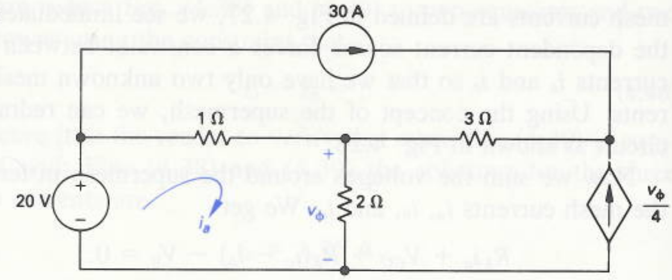
DRILL EXERCISES

4.14 Use the mesh-current method to find the power dissipated in the 2-Ω resistor.



ANSWER: 72 W.

4.15 Use the mesh-current method to find the mesh current i_a in the following circuit.



ANSWER: 10 A.

4.8 THE NODE-VOLTAGE METHOD VERSUS THE MESH-CURRENT METHOD

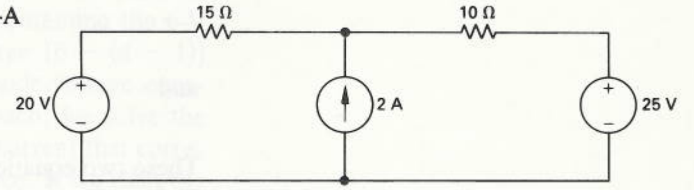
The greatest advantage of both the node-voltage and mesh-current methods is that they reduce the number of simultaneous equations that must be manipulated. They also require the analyst to be quite systematic in organizing and writing the required simultaneous equations. It is natural to ask then, "When is the node-voltage method preferred to the mesh-current method, and vice versa?" As one might suspect, there is no clear-cut answer. One possible approach is to compare the number of simultaneous equations required for each method and then to select the one requiring the least number. A second is to analyze the presence and location of voltage and current sources within the circuit structure. Voltage sources may require extra effort in formulating node-voltage equations, whereas current sources may require extra effort in formulating mesh-current equations.

Another point to consider when choosing between the two methods is what information about the circuit being analyzed is of primary interest. In other words, a complete solution of a circuit may not be needed, and therefore the particular piece of information that is of interest may influence what method is used. For example, in the circuit in Fig. 2.18 if only i_{CC} is of interest, the mesh-current method may be selected, whereas if only the voltage across R_2 is of interest, the node-voltage method might be favored.

Perhaps the most important observation to make regarding these two methods of circuit analysis is that in any given situation a little time spent thinking about the problem in relation to the various analytical approaches available will be time well spent.

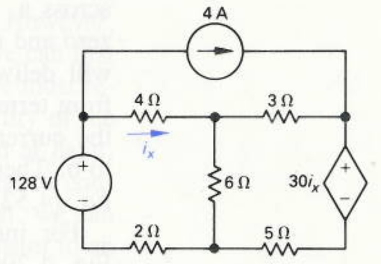
DRILL EXERCISES

4.16 Find the power delivered to the circuit by a 2-A current source.



ANSWER: 70 W.

4.17 Find the power delivered to the circuit by the 4-A current source.



ANSWER: 40 W.

4.9 SOURCE TRANSFORMATIONS

Even though the node-voltage and mesh-current methods are powerful techniques for solving circuits, we are still interested in methods that can be used to simplify circuits. We begin expanding our list of simplifying techniques with source transformations. A *source transformation,* shown in Fig. 4.29, allows us to replace a voltage source in series with a resistor by a current source in parallel with the same resistor, or vice versa. The double-headed arrow in Fig. 4.29 is used to emphasize that a source transformation is bilateral, that is, we can start with either configuration and derive the other. The two configurations

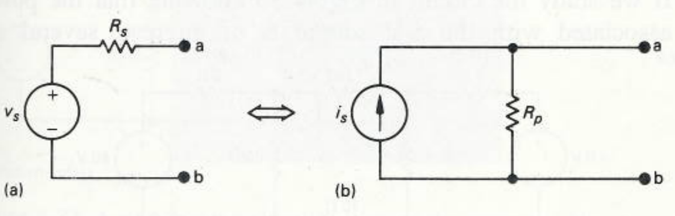


Figure 4.29 Source transformations.

shown in Fig. 4.29 are equivalent with respect to the terminals a, b provided

$$i_s = \frac{v_s}{R_s} \tag{4.52}$$

and

$$R_s = R_p. \tag{4.53}$$

These two equations can be verified by the following arguments. If the two circuits are equivalent with respect to the terminals a, b, they must be equivalent for *all* external values of R connected across a, b. Two extreme values of R that are easy to test are zero and infinity. For 0 Ω, or a short circuit, the voltage source will deliver a short-circuit current of v_s/R_s amperes, oriented from terminal a toward b. The short-circuit current delivered by the current source would be i_s, also oriented from terminal a to b. These two short-circuit currents are identical by virtue of Eq. (4.52).

For infinite external resistance, the source arrangement of Fig. 4.29(a) predicts that the voltage from a to b would be v_s with terminal a positive. The voltage across a, b in the circuit in Fig. 4.29(b) is $i_s R_p$, which is equal to v_s by virtue of Eqs. (4.52) and (4.53). Terminal a is also positive, as it must be in order for the two source arrangements to be equivalent.

Observe that if the polarity of v_s is reversed, the orientation of i_s must be reversed in order to maintain equivalence.

The usefulness of making source transformations in order to simplify a circuit-analysis problem is illustrated by the following example.

EXAMPLE 4.5

Source transformations

a) For the circuit shown in Fig. 4.30, find the power associated with the 6-V source.
b) State whether the 6-V source is absorbing or delivering the power calculated in part (a).

SOLUTION

a) If we study the circuit in Fig. 4.30 knowing that the power associated with the 6-V source is of interest, several ap-

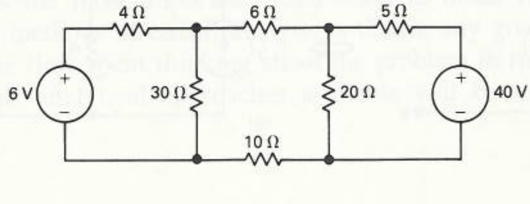


Figure 4.30 The circuit for Example 4.5.

proaches come to mind. First we note that the circuit has four essential nodes and six essential branches where the current is unknown. Thus the current in the branch containing the 6-V source can be found by solving either three [6 − (4 − 1)] mesh-current equations or three (4 − 1) node-voltage equations. If we choose the mesh-current approach, we solve the three mesh-current equations for the mesh current that corresponds to the branch current in the 6-V source. If we elect the node-voltage approach, we solve the three node-voltage equations for the voltage across the 30-Ω resistor, from which the branch current in the 6-V source can be calculated. However, since we are focusing on just one branch current, we can first simplify the circuit using source transformations. We must reduce the circuit in a manner that preserves the identity of the branch containing the 6-V source. For the problem at hand, there is no reason to preserve the identity of the branch containing the 40-V source. Beginning with this branch, we can transform the 40-V source in series with the 5-Ω resistor to an 8-A current source in parallel with a 5-Ω resistor, as shown in Fig. 4.31(a). Next, the parallel combination of the 20-Ω and 5-Ω resistors can be replaced with a 4-Ω resistor. This 4-Ω resistor shunts the 8-A source and therefore can be replaced by a 32-V source in series with a 4-Ω resistor, as shown in Fig. 4.31(b). The 32-V source is in series with 20 Ω of resistance and, hence, can be replaced by a current source of 1.6 A in parallel with 20 Ω, as shown in Fig. 4.31(c). The parallel combination of the 1.6-A current source and the

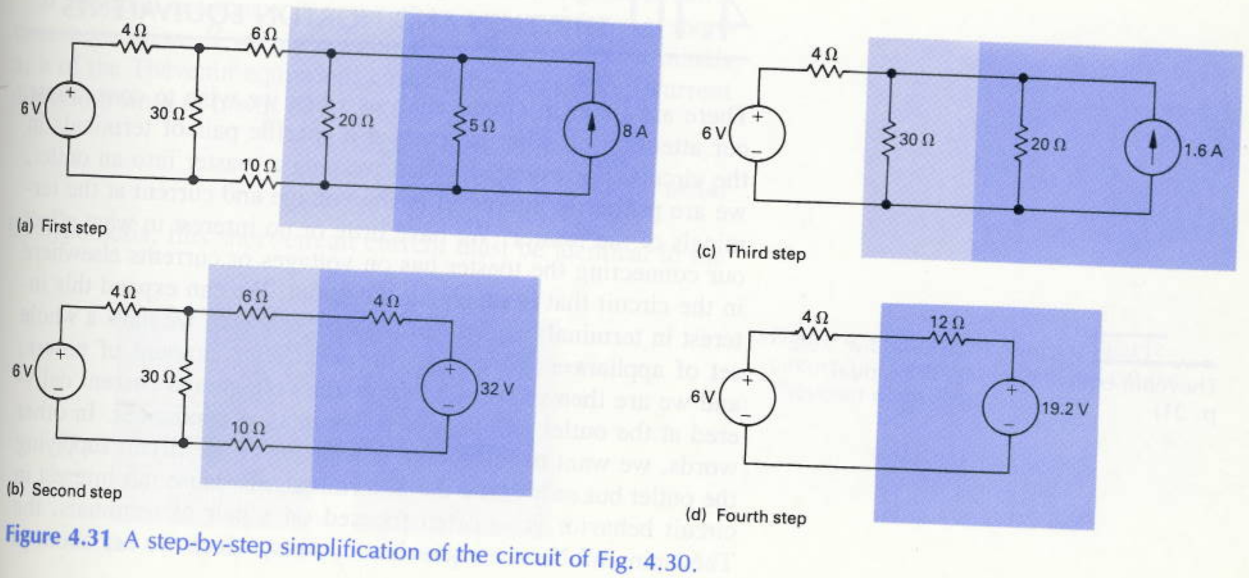


Figure 4.31 A step-by-step simplification of the circuit of Fig. 4.30.

12-Ω resistor transforms to a voltage source of 19.2 V in series with 12 Ω. The result of this last transformation is shown in Fig. 4.31(d), from which we see that the current in the direction of the voltage drop across the 6-V source is (19.2 − 6)/16, or 0.825 A. Therefore the power associated with the 6-V source is

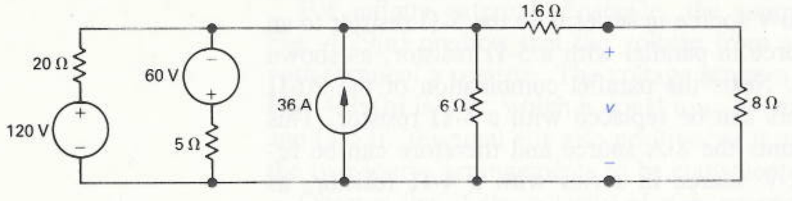
$$p_{6\text{V}} = (0.825)(6) = 4.95 \text{ W}.$$

b) The voltage source is absorbing power.

DRILL EXERCISES

4.18 a) Use a series of source transformations to find the voltage v in the circuit shown.

b) How much power does the 120-V source deliver to the circuit?



ANSWER: (a) 48 V; (b) 374.4 W.

4.10 THÉVENIN AND NORTON EQUIVALENTS

SPICE Using SPICE to find a Thévenin equivalent: Sec. 6 (manual p. 21)

There are times in circuit analysis when we wish to concentrate our attention on what happens at a specific pair of terminals in the circuit. For example, when we plug a toaster into an outlet, we are primarily interested in the voltage and current at the terminals of the toaster. We have little or no interest in what effect our connecting the toaster has on voltages or currents elsewhere in the circuit that is supplying the outlet. We can expand this interest in terminal behavior to the case in which we have a whole set of appliances, each requiring a different amount of power, and we are then interested in how the voltage and current delivered at the outlet will change as we change appliances. In other words, we want to focus on the behavior of the circuit supplying the outlet but only at the outlet terminals. Because our interest in circuit behavior is so often focused on a pair of terminals, the Thévenin and Norton equivalent circuits, which we are about to introduce, are extremely valuable aids in analysis. Although at this time we will discuss these equivalent circuits as they pertain to resistive circuits, you should be aware, right at the outset, that Thévenin and Norton equivalent circuits can be used to represent any circuit made up of linear elements.

The significance of the Thévenin equivalent circuit can be described using Fig. 4.32. Figure 4.32(a) represents any circuit made up of sources (both independent and dependent) and resistors. We have identified by the letters a and b the pair of terminals that are of interest. In Fig. 4.32(b), we show the Thévenin equivalent. What we imply by the circuit in Fig. 4.32(b) is that the original interconnection of sources and resistors can be replaced by an independent voltage source V_t in series with a resistor R_t. Furthermore, this series combination of V_t and R_t is equivalent to the original circuit in the sense that if we connect the same load across the terminals a, b of each circuit, we get the same voltage and current at the terminals of the load. This equivalence will hold for *all possible values of load resistance*.

In order to represent the original circuit by its Thévenin equivalent, we must be able to determine the Thévenin voltage V_t and the Thévenin resistance R_t. These two parameters of the Thévenin equivalent can be found as follows. First we note that if the load resistance is infinitely large, we have an open-circuit condition. The open-circuit voltage at the terminals a, b in the circuit in Fig. 4.32(b) will be V_t. By hypothesis, this must be the same as the open-circuit voltage at the terminals a, b in the original circuit. Therefore to calculate the Thévenin voltage V_t, we simply calculate the open-circuit voltage in the original circuit.

Thévenin voltage

If the load resistance is reduced to zero, we have a short-circuit condition. If we place a short circuit across the terminals a, b of the Thévenin equivalent circuit, the short-circuit current directed from a to b will be

$$i_{sc} = \frac{V_t}{R_t}. \quad (4.54)$$

By hypothesis, this short-circuit current must be identical to the

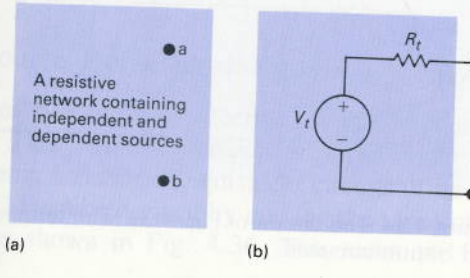


Figure 4.32 A Thévenin equivalent circuit: (a) a general circuit and (b) the Thévenin equivalent.

short-circuit current that exists in a short circuit placed across the terminals a, b of the original network. From Eq. (4.54) we have

$$R_t = \frac{V_t}{i_{sc}}. \tag{4.55}$$

Thus the Thévenin resistance is the ratio of the open-circuit voltage to the short-circuit current. Let us demonstrate with a specific current.

Finding a Thévenin Equivalent

To find the Thévenin equivalent circuit of the circuit shown in Fig. 4.33, we first calculate the open-circuit voltage of v_{ab}. Note that when the terminals a, b are open, there will be no current in the 4-Ω resistor. Therefore the open-circuit voltage v_{ab} will be identical to the voltage across the 3-A current source. This voltage has been labeled v_0 on the circuit in Fig. 4.33. The voltage v_0 can be found by solving a single node-voltage equation. Choosing the lower node as the reference node, we have

$$\frac{v_0 - 25}{5} + \frac{v_0}{20} - 3 = 0. \tag{4.56}$$

Solving for v_0 yields

$$v_0 = 32 \text{ V}. \tag{4.57}$$

It follows that the Thévenin voltage for the circuit in Fig. 4.33 is 32 V.

The next step in deriving the Thévenin equivalent circuit with respect to the terminals a, b is to place a short circuit across the terminals and calculate the resulting short-circuit current. The circuit with the short in place is shown in Fig. 4.34. Note that the short-circuit current is in the direction of the open-circuit voltage drop across the terminals a, b. [If the short-circuit current is in the direction of the open-circuit voltage rise across the terminals, a minus sign must be inserted in Eq. (4.55).]

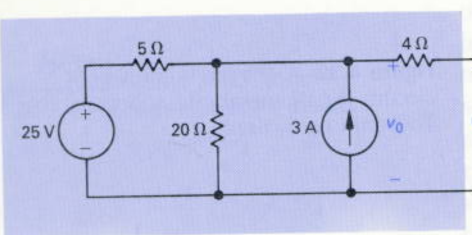


Figure 4.33 A circuit used to illustrate a Thévenin equivalent.

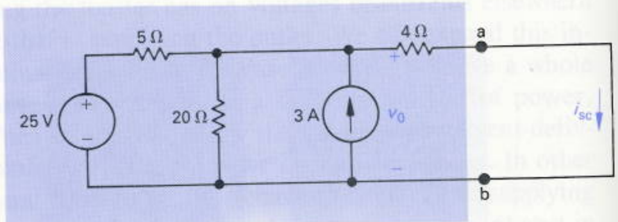


Figure 4.34 The circuit of Fig. 4.33 with terminals a and b short-circuited.

The short-circuit current (i_{sc}) is easily found once v_0 is known. Therefore the problem reduces to finding v_0 with the short in place. Again, if we use the lower node as the reference node, the equation for v_0 becomes

$$\frac{v_0 - 25}{5} + \frac{v_0}{20} - 3 + \frac{v_0}{4} = 0. \quad (4.58)$$

Solving Eq. (4.58) for v_0 gives us

$$v_0 = 16 \text{ V}. \quad (4.59)$$

It follows that the short-circuit current is

$$i_{sc} = \frac{16}{4} = 4 \text{ A}. \quad (4.60)$$

Now we can find the Thévenin resistance by substituting the numerical values given in Eqs. (4.57) and (4.60) into Eq. (4.55). Thus we have

$$R_t = \frac{V_t}{i_{sc}} = \frac{32}{4} = 8\ \Omega. \quad (4.61)$$

The Thévenin equivalent circuit for the circuit in Fig. 4.33 is shown in Fig. 4.35.

We will leave it to you to verify that if a 24-Ω resistor is connected across the terminals a, b in the circuit shown in Fig. 4.33, the voltage across the resistor will be 24 V and the current in the resistor will be 1 A. We can see by inspection that the Thévenin circuit in Fig. 4.35 will predict the same voltage and current if a 24-Ω resistor is connected across the terminals a, b.

The Norton Equivalent

The Norton equivalent circuit consists of an independent current source in parallel with the Norton equivalent resistance. It can be derived from the Thévenin equivalent circuit simply by making a source transformation. Thus the Norton current equals the short-circuit current at the terminals of interest, and the Norton resistance is identical to the Thévenin resistance.

Using Source Transformations

Sometimes we can make effective use of source transformations to derive a Thévenin or Norton equivalent circuit. For example, the Thévenin and Norton equivalent circuits of the circuit shown in Fig. 4.33 can be derived by making the series of source transformations shown in Fig. 4.36. This technique is most useful

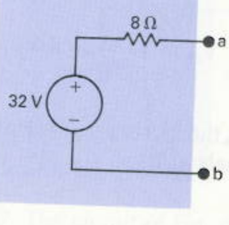


Figure 4.35 The Thévenin equivalent of the circuit shown in Fig. 4.33.

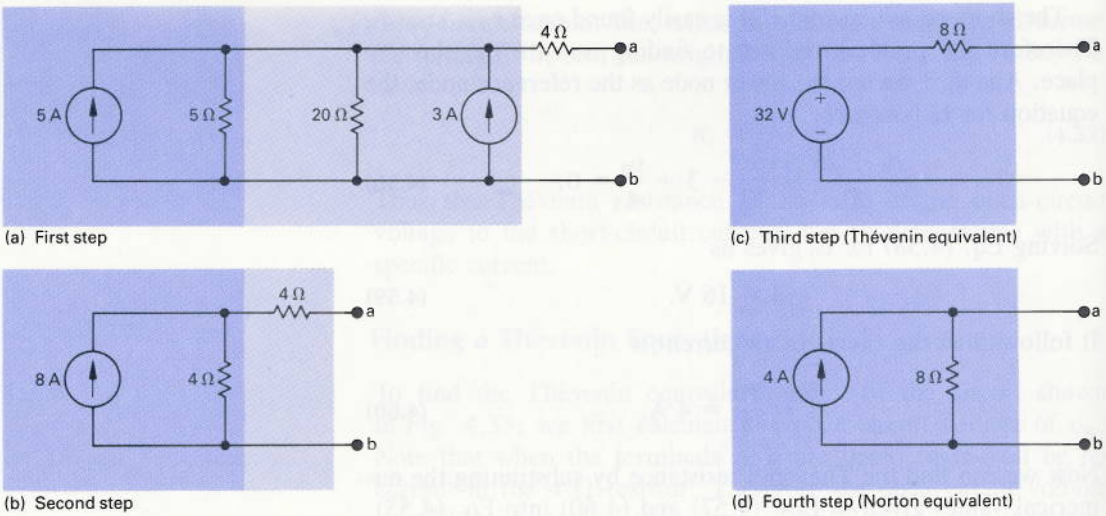
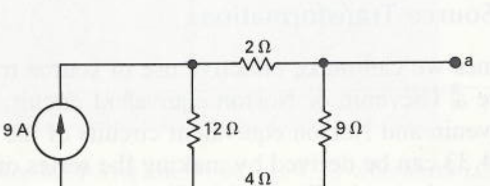


Figure 4.36 The step-by-step derivation of the Thévenin and Norton equivalent circuits of the circuit shown in Fig. 4.33.

when the network contains only independent sources. The presence of dependent sources will require retaining the identity of the controlling voltages and/or currents, and this constraint will usually prohibit the continued reduction of the circuit via source transformations. In Section 4.11 we will illustrate the problem of finding the Thévenin equivalent when the circuit contains dependent sources.

DRILL EXERCISES

4.19 Find the Thévenin equivalent circuit with respect to the terminals a, b.

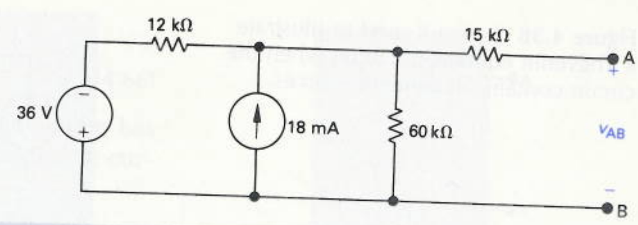


ANSWER: $V_{ab} = V_t = 64.8$ V, $R_t = 6\ \Omega$.

4.20 Find the Norton equivalent circuit with respect to the terminals a, b.

ANSWER: $I_n = 6$ A (directed toward a), $R_n = 6\ \Omega$.

4.21 A voltmeter with an internal resistance of 100 kΩ is used to measure the voltage v_{AB} in the circuit shown. What is the voltmeter reading?



ANSWER: 120 V.

4.11 MORE ON DERIVING A THÉVENIN EQUIVALENT

The technique for determining R_t that we discussed and illustrated in the preceding section is not always the easiest method. There are two other methods that are generally simpler to implement. The first is useful only if the network contains independent sources. To calculate R_t for such a network, we first deactivate all independent sources and then calculate the resistance seen looking into the network at the designated terminal pair. *A voltage source is deactivated by replacing it with a short circuit. A current source is deactivated by replacing it with an open circuit*. As an example, consider the circuit shown in Fig. 4.33. Once the independent sources have been deactivated, the circuit in Fig. 4.33 simplifies to that shown in Fig. 4.37. The resistance seen looking into the terminals a, b is denoted by R_{ab} in Fig. 4.37. We can see from this circuit that R_{ab} consists of the 4-Ω resistor in series with the parallel combinations of the 5- and 20-Ω resistors; thus

$$R_{ab} = R_t = 4 + \frac{5 \times 20}{25} = 8\ \Omega. \quad \textbf{(4.62)}$$

Note that the derivation of R_t via Eq. (4.62) is much simpler than the derivation of R_t via Eq. (4.61).

If the network contains dependent sources, an alternative procedure for finding the Thévenin resistance R_t is as follows (see Fig. 4.38). We first deactivate all *independent* sources. We then apply either a test voltage source or a test current source to the Thévenin terminals a, b. The Thévenin resistance will equal the ratio of the voltage across the test source to the current delivered by the test source. In studying the circuit shown in Fig. 4.38, note that it contains an independent 5-V source, a voltage-controlled voltage source, and a current-controlled current source. Also note the controlling signals. The dependent voltage source is controlled by the voltage across the 25-Ω resistor, and the dependent current source is controlled by the current in the

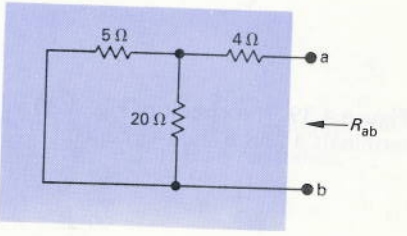


Figure 4.37 The circuit of Fig. 4.33 after the independent sources have been deactivated.

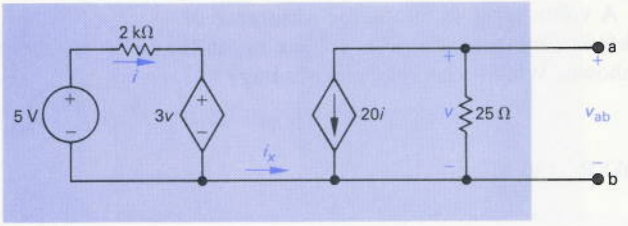


Figure 4.38 A circuit used to illustrate a Thévenin equivalent circuit when the circuit contains dependent sources.

2-kΩ resistor. We will first derive the Thévenin equivalent circuit with respect to the terminals a, b using the open-circuit voltage and short-circuit current calculations. We will then illustrate the alternative method for finding the Thévenin resistance.

Dependent Sources—A Numerical Example

The first step in analyzing the circuit in Fig. 4.38 is to recognize that the current labeled i_x must be zero. (Note that there is no return path for i_x to enter the left-hand portion of the circuit.) The open-circuit, or Thévenin, voltage will be the voltage across the 25-Ω resistor. Since i_x is zero, it follows directly that

$$V_t = v_{ab} = (-20i)(25) = -500i. \qquad \textbf{(4.63)}$$

Now we calculate the current i:

$$i = \frac{5 - 3v}{2000} = \frac{5 - 3V_t}{2000}. \qquad \textbf{(4.64)}$$

In writing Eq. (4.64), we recognize that the Thévenin voltage is identical to the control voltage. When we substitute Eq. (4.64) into Eq. (4.63), we find that

$$V_t = -5 \text{ V}. \qquad \textbf{(4.65)}$$

To calculate the short-circuit current, we place a short circuit across a, b. Observe that when the terminals a, b are shorted together, the control voltage v is reduced to zero. Therefore, with the short in place, the circuit in Fig. 4.38 becomes as shown in Fig. 4.39. With the short circuit shunting the 25-Ω resistor, all the current from the dependent current source will appear in the

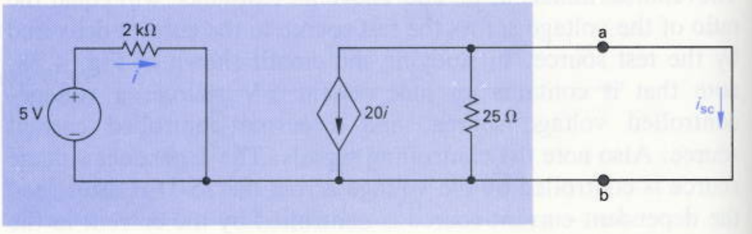


Figure 4.39 The circuit of Fig. 4.38 with terminals a and b short-circuited.

short; thus

$$i_{sc} = -20i. \quad \textbf{(4.66)}$$

Since the voltage controlling the dependent voltage source has been reduced to zero, the current controlling the dependent current source is

$$i = \frac{5}{2000} = 2.5 \text{ mA}. \quad \textbf{(4.67)}$$

When Eq. (4.67) is substituted into Eq. (4.66), we get a short-circuit current of

$$i_{sc} = -20(2.5) = -50 \text{ mA}. \quad \textbf{(4.68)}$$

It follows directly from Eqs. (4.65) and (4.68) that

$$R_t = \frac{V_t}{i_{sc}} = \frac{-5}{-50} \times 10^3 = 100\ \Omega. \quad \textbf{(4.69)}$$

The Thévenin equivalent circuit for the circuit shown in Fig. 4.38 is illustrated in Fig. 4.40. Note that the reference polarity marks on the Thévenin voltage source in Fig. 4.40 agree with Eq. (4.65).

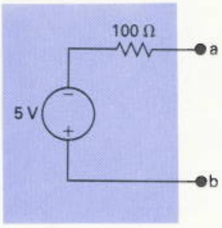


Figure 4.40 The Thévenin equivalent for the circuit shown in Fig. 4.38.

Dependent Sources—The Thévenin Resistance by an Alternate Method

Let us now consider the alternative technique for finding the Thévenin resistance R_t. We first deactivate the independent voltage source from the circuit and then excite the circuit from the terminals a, b with either a test voltage source or a test current source. In deciding which type of source to use, we note from the circuit that if we apply a test voltage source we will know the voltage of the dependent voltage source and, hence, the controlling current i. Therefore in this circuit we opt for the test voltage source. The circuit for computing the Thévenin resistance is shown in Fig. 4.41. The externally applied test voltage source is denoted by v_T and the current that it delivers to the circuit is labeled i_T. To find the Thévenin resistance, we simply solve the circuit shown in Fig. 4.41 for the ratio of the

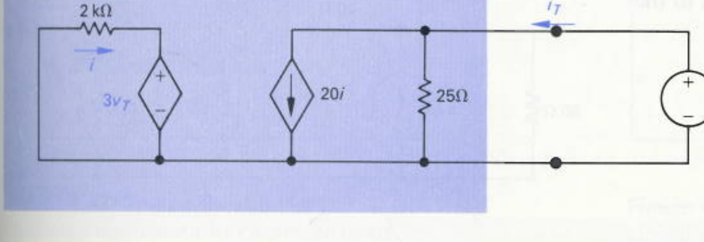


Figure 4.41 An alternative method for computing the Thévenin resistance.

voltage to the current at the test source, that is, $R_t = v_T/i_T$. From Fig. 4.41 we note that

$$i_T = \frac{v_T}{25} + 20i \tag{4.70}$$

and

$$i = \frac{-3v_T}{2}\text{mA}. \tag{4.71}$$

We then substitute Eq. (4.71) into Eq. (4.70) and solve the resulting equation for the ratio v_T/i_T:

$$i_T = \frac{v_T}{25} - \frac{60v_T}{2000},$$

$$\frac{i_T}{v_T} = \frac{1}{25} - \frac{6}{200} = \frac{50}{5000} = \frac{1}{100}. \tag{4.72}$$

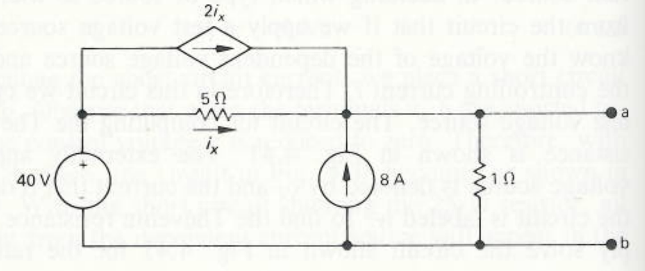
It follows from Eqs. (4.72) that

$$R_t = \frac{v_T}{i_T} = 100\ \Omega. \tag{4.73}$$

In general, the computations involved in this alternative method for finding the Thévenin resistance are easier than those involved in computing the short-circuit current. We should also point out that in a network containing only resistors and dependent sources, the alternative method must be used because the ratio of the Thévenin voltage to the short-circuit current is indeterminate; that is, it is the ratio 0/0. (See Problem 4.29.)

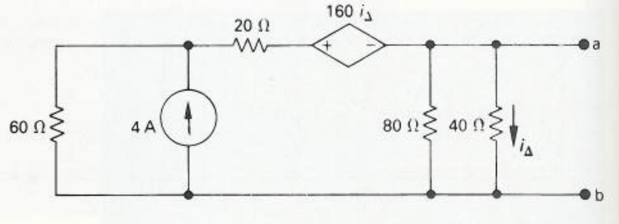
DRILL EXERCISES

4.22 Find the Thévenin equivalent circuit with respect to the terminals a, b.



ANSWER: $V_t = v_{ab} = 20$ V, $R_t = 0.625\ \Omega$.

4.23 Find the Thévenin equivalent with respect to the terminals a, b.



ANSWER: $V_t = v_{ab} = 30$ V, $R_t = 10\ \Omega$.

Illustration of a Useful Application

There are times when a Thévenin equivalent can be used to reduce one portion of a larger circuit so that the analysis of the larger network is greatly simplified. Let us return to the circuit first introduced in Section 2.5 and subsequently analyzed in Sections 4.4 and 4.7. To facilitate our discussion of using Thévenin's theorem to analyze this circuit, we have redrawn the circuit in Fig. 4.42 and identified the branch currents of interest. Before using a Thévenin equivalent to effect a solution, we make the observation that once we know i_B, we can easily obtain the other branch currents. We argue as follows. The current i_E is simply $(1 + \beta)i_B$. When i_E is known, the voltages v_{cd} and, hence, v_{bd} are known since $v_{\mathrm{bd}} = v_{\mathrm{cd}} + V_0$. When we know the voltage v_{bd}, we can quickly compute the branch currents i_1 and i_2. Thus $i_2 = v_{\mathrm{bd}}/R_2$ and $i_1 = i_2 + i_B$. Realizing that i_B is the key to finding the other branch currents, we redraw the circuit as shown in Fig. 4.43. With a little thought, you should be able to observe that this modification will have no effect on the branch currents i_1, i_2, i_B, and i_E.

Now we replace the circuit made up of V_{CC}, R_1, and R_2 with a Thévenin equivalent. The equivalent is made with respect to the terminals b, d. The Thévenin voltage and resistance are

$$V_t = \frac{V_{CC}R_2}{R_1 + R_2} \tag{4.74}$$

and

$$R_t = \frac{R_1R_2}{R_1 + R_2}. \tag{4.75}$$

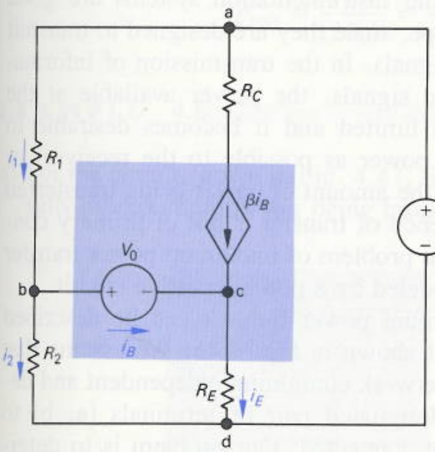


Figure 4.42 The application of a Thévenin equivalent in circuit analysis.

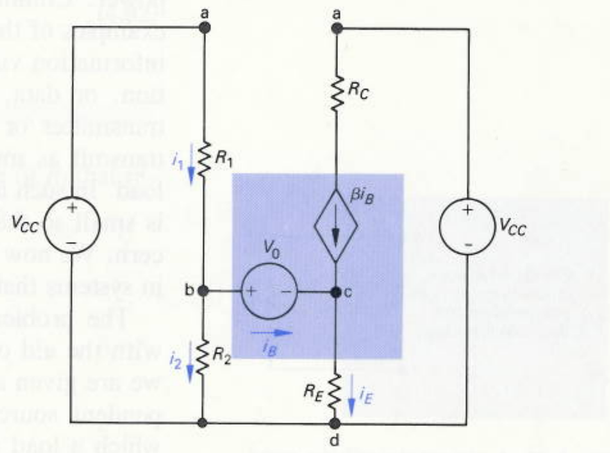


Figure 4.43 A modified version of the circuit shown in Fig. 4.42.

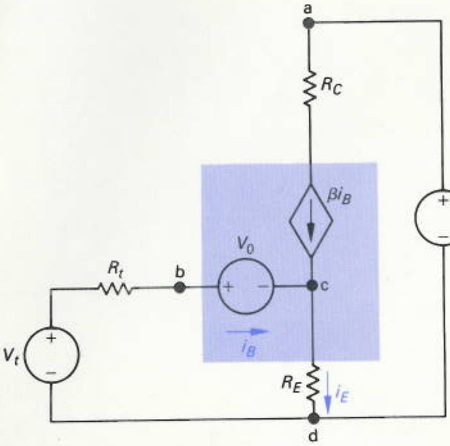


Figure 4.44 The circuit of Fig. 4.43 modified by a Thévenin equivalent.

With the Thévenin equivalent, the circuit in Fig. 4.43 becomes as shown in Fig. 4.44.

We can now derive an equation for i_B simply by summing the voltages around the left mesh. In writing this mesh equation, we use the fact that $i_E = (1 + \beta)i_B$. Thus

$$V_t = R_t i_B + V_0 + R_E(1 + \beta)i_B, \tag{4.76}$$

from which we can write

$$i_B = \frac{V_t - V_0}{R_t + (1 + \beta)R_E}. \tag{4.77}$$

When we substitute Eqs. (4.74) and (4.75) into Eq. (4.77), we obtain the same expression denoted by Eq. (2.27). Note that once we have incorporated the Thévenin equivalent into the original circuit, we can obtain the solution for i_B by writing a single equation!

4.12 MAXIMUM POWER TRANSFER

Circuit analysis plays an important role in the analysis of systems that are designed to transfer power from a source to a load. The general problem of power transfer can be discussed in terms of two basic types of systems. One emphasizes the efficiency of the power transfer, and the other emphasizes the amount of the power transfer. Power utility systems are a good example of the first type since they are concerned with the generation, transmission, and distribution of large quantities of electric power. Communication and instrumentation systems are good examples of the second type, since they are designed to transmit information via electric signals. In the transmission of information, or data, via electric signals, the power available at the transmitter or detector is limited and it becomes desirable to transmit as much of this power as possible to the receiver, or load. In such applications the amount of power being transferred is small so that the efficiency of transfer is not of primary concern. We now consider the problem of maximum power transfer in systems that can be modeled by a purely resistive circuit.

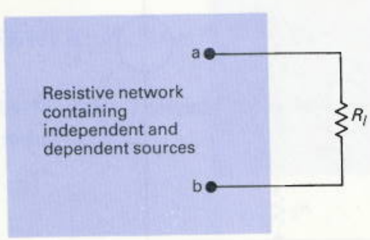


Figure 4.45 A circuit describing maximum power transfer.

The problem of maximum power transfer can be described with the aid of the circuit shown in Fig. 4.45. We assume that we are given a resistive network containing independent and dependent sources and a designated pair of terminals (a, b) to which a load (R_l) is to be connected. Our problem is to determine the value of R_l such that maximum power will be delivered to R_l. The first step to finding the critical value of R_l is to recognize that the given resistive network can always be replaced by its Thévenin equivalent. Therefore we redraw the circuit in Fig. 4.45 as shown in Fig. 4.46. Once we have replaced the original network by its Thévenin equivalent, we have greatly simplified the problem of finding R_l. The derivation for R_l requires expressing the power dissipated in R_l as a function of the three circuit parameters V_t, R_t, and R_l. Thus

$$p = i^2 R_l = \left(\frac{V_t}{R_t + R_l}\right)^2 R_l. \quad \textbf{(4.78)}$$

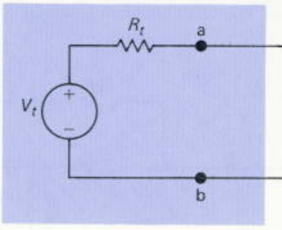


Figure 4.46 A circuit used to detemine the value of R_l for maximum power transfer.

Next, we recognize that for a given circuit, V_t and R_t will be fixed; therefore the power dissipated is a function of the single variable R_l. To find the value of R_l that maximizes the power, we use elementary calculus; that is, we solve for the value of R_l where dp/dR_l equals zero. We have

$$\frac{dp}{dR_l} = V_t^2\left[\frac{(R_t + R_l)^2 - R_l \cdot 2(R_t + R_l)}{(R_t + R_l)^4}\right]. \quad \textbf{(4.79)}$$

Now the derivative will be zero when

$$(R_t + R_l)^2 = 2R_l(R_t + R_l). \quad \textbf{(4.80)}$$

Solving Eq. (4.80) yields

$$R_l = R_t. \quad \textbf{(4.81)}$$

Thus maximum power transfer occurs when the load resistance R_l equals the Thévenin resistance R_t. To find the maximum power delivered to R_l, we simply substitute Eq. (4.81) into Eq. (4.78) to get

$$p_{\max} = \frac{V_t^2 R_l}{(2R_l)^2} = \frac{V_t^2}{4R_l}. \quad \textbf{(4.82)}$$

EXAMPLE 4.6

a) For the circuit shown in Fig. 4.47, find the value of R_l that results in maximum power being transferred to R_l.

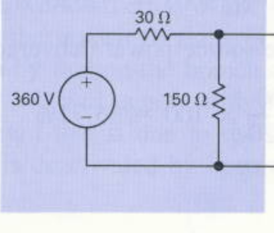


Figure 4.47 The circuit for Example 4.6.

b) Calculate the maximum power that can be delivered to R_l.

c) When R_l is adjusted for maximum power transfer, what percentate of the power delivered by the 360-V source reaches R_l?

SOLUTION

a) The Thévenin voltage for the circuit to the left of the terminals a, b is

$$V_t = \frac{360}{180} \times 150 = 300 \text{ V}.$$

The Thévenin resistance is

$$R_t = \frac{(150)(30)}{180} = 25\ \Omega.$$

Replacing the circuit to the left of the terminals a, b with its Thévenin equivalent gives us the circuit shown in Fig. 4.48, from which we see that R_l must equal 25 Ω for maximum power transfer.

b) The maximum power that can be delivered to R_l is

$$p_{\max} = \left(\frac{300}{50}\right)^2 (25) = 900 \text{ W}.$$

c) When R_l equals 25 Ω, the voltage v_{ab} is

$$v_{ab} = \left(\frac{300}{50}\right)(25) = 150 \text{ V}.$$

We can see from Fig. 4.47 that when v_{ab} equals 150 V, the current in the voltage source in the direction of the voltage rise across the source will be

$$i_s = \frac{360 - 150}{30} = \frac{210}{30} = 7 \text{ A}.$$

Therefore the source is delivering 2520 W to the circuit, that is,

$$p_s = -i_s(360) = -2520 \text{ W}.$$

The percentage of the source power delivered to the load is

$$\frac{900}{2520} \times 100 = 35.71\%.$$

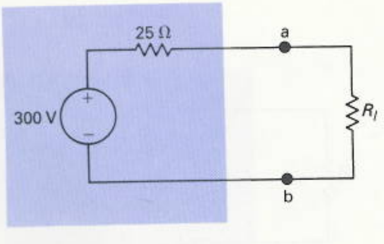
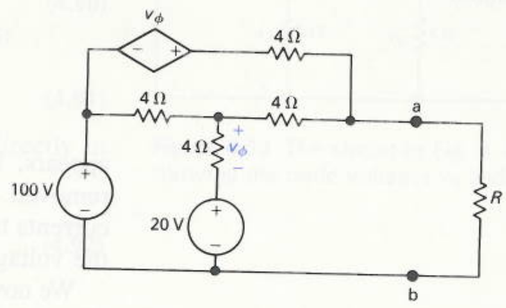


Figure 4.48 A reduction of the circuit of Fig. 4.47 by means of a Thévenin equivalent.

DRILL EXERCISES

4.24 a) Find the value of R that will enable the circuit to deliver maximum power to the terminals a, b.

b) Find the maximum power delivered to R.



ANSWER: (a) 3 Ω; (b) 1.2 kW.

4.25 Assume that the circuit in Drill Exercise 4.24 is delivering maximum power to the load resistor R.

a) How much power is the 100-V source delivering to the network?

b) Repeat part (a) for the dependent voltage source.

c) What percentage of the total power generated by these two sources is delivered to the load resistor R?

ANSWER: (a) 3000 W; (b) 800 W; (c) 31.58%.

4.13 SUPERPOSITION

The most distinguishing characteristic of a linear system is the principle of *superposition,* which states that whenever a linear system is excited, or driven, by more than one independent source of energy, we can find the total response by finding the response to each independent source separately and then summing the individual responses. Since we are dealing with circuits made up of interconnected linear-circuit elements, we can apply the principle of superposition directly to the analysis of such circuits when they are driven by more than one independent energy source. At present, we will restrict our illustration to simple resistive networks; however, it is important to bear in mind that the principle is applicable to circuits containing inductance and capacitance as well as resistance. As a matter of fact, it is applicable to any linear system.

We will demonstrate the use of the superposition principle by using it to find the branch currents in the circuit shown in Fig. 4.49. We begin by finding the branch currents due to the 120-V voltage source. We will denote with a prime the component of the branch current that is due to the voltage source. The ideal current source is deactivated by opening the branch in which it

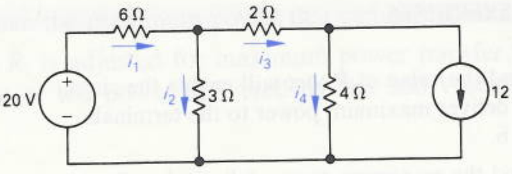


Figure 4.49 A circuit used to illustrate superposition.

appears. Figure 4.50 shows the circuit with the current source removed. We see there the prime notation applied to the branch currents to indicate that the currents in the circuit are due only to the voltage source.

We now note that we can easily find the branch currents in the circuit in Fig. 4.50 once we know the node voltage across the 3-Ω resistor. If we denote this voltage as v_1, we can write

$$\frac{v_1 - 120}{6} + \frac{v_1}{3} + \frac{v_1}{2+4} = 0, \tag{4.83}$$

from which it follows that

$$v_1 = 30 \text{ V}. \tag{4.84}$$

Now we can write the expressions for the branch currents i_1' through i_4' directly; thus

$$i_1' = \frac{120 - 30}{6} = 15 \text{ A}, \tag{4.85}$$

$$i_2' = \frac{30}{3} = 10 \text{ A}, \tag{4.86}$$

$$i_3' = i_4' = \frac{30}{6} = 5 \text{ A}. \tag{4.87}$$

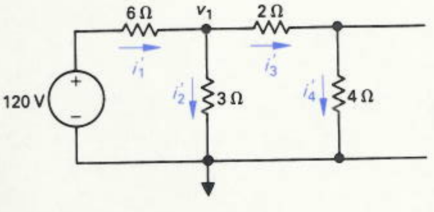


Figure 4.50 The circuit of Fig. 4.49 with the current source deactivated.

The ideal voltage source in Fig. 4.49 is deactivated by replacing it with a short circuit. Thus to find the component of the branch currents due to the current source, we must solve the circuit shown in Fig. 4.51. The double-prime notation for the currents in Fig. 4.51 indicates that these currents are the components of the total current due to the ideal current source.

We will solve for the branch currents in the circuit in Fig. 4.51 by first solving for the node voltages across the 3- and 4-Ω resistors, respectively. The two node voltages are defined as shown in Fig. 4.52, from which it follows that the two node-voltage equations that describe the circuit are

$$\frac{v_3}{3} + \frac{v_3}{6} + \frac{v_3 - v_4}{2} = 0, \tag{4.88}$$

$$\frac{v_4 - v_3}{2} + \frac{v_4}{4} + 12 = 0. \tag{4.89}$$

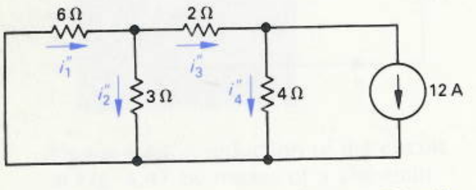


Figure 4.51 The circuit of Fig. 4.49 with the voltage source deactivated.

When we solve Eqs. (4.88) and (4.89) for v_3 and v_4 we get

$$v_3 = -12 \text{ V} \tag{4.90}$$

and

$$v_4 = -24 \text{ V} \tag{4.91}$$

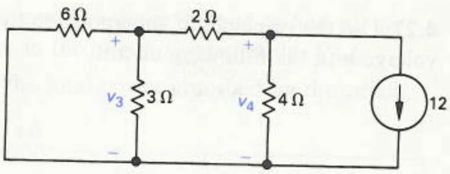


Figure 4.52 The circuit of Fig. 4.49 showing the node voltages v_3 and v_4.

Now we can write the branch currents i_1'' through i_4'' directly in terms of the node voltages v_3 and v_4 as follows:

$$i_1'' = \frac{-v_3}{6} = \frac{12}{6} = 2 \text{ A}, \tag{4.92}$$

$$i_2'' = \frac{v_3}{3} = \frac{-12}{3} = -4 \text{ A}, \tag{4.93}$$

$$i_3'' = \frac{v_3 - v_4}{2} = \frac{-12 + 24}{2} = 6 \text{ A}, \tag{4.94}$$

$$i_4'' = \frac{v_4}{4} = \frac{-24}{4} = -6 \text{ A}. \tag{4.95}$$

To find the branch currents in the original circuit—that is, the currents i_1, i_2, i_3, and i_4 in the circuit shown in Fig. 4.49—we simply add the currents given by Eqs. (4.92)–(4.95) to the currents given by Eqs. (4.85)–(4.87). Thus

$$i_1 = i_1' + i_1'' = 15 + 2 = 17 \text{ A}, \tag{4.96}$$

$$i_2 = i_2' + i_2'' = 10 - 4 = 6 \text{ A}, \tag{4.97}$$

$$i_3 = i_3' + i_3'' = 5 + 6 = 11 \text{ A}, \tag{4.98}$$

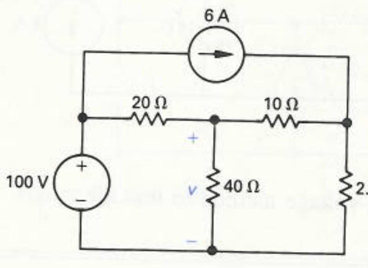
$$i_4 = i_4' + i_4'' = 5 - 6 = -1 \text{ A}. \tag{4.99}$$

We leave it to the reader to verify that the currents given by Eqs. (4.96)–(4.99) are the correct values for the branch currents in the circuit in Fig. 4.49.

DRILL EXERCISES

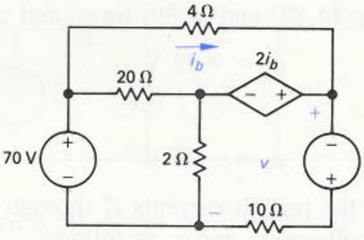
4.26 a) Use the principle of superposition to find the voltage v in the circuit shown.

b) Find the power dissipated in the 40-Ω resistor.



ANSWER: (a) 40 V; (b) 40 W.

4.27 Use the principle of superposition to find the voltage v in the following circuit.



ANSWER: 30 V.

SUMMARY

The purpose of this chapter has been to introduce some extremely useful techniques of circuit analysis. The node-voltage and mesh-current techniques are important because they enable us to minimize the number of simultaneous equations needed to describe a circuit. These two techniques are also valuable because they force us to take a systematic approach to writing circuit equations. Source transformations and Thévenin–Norton equivalent circuits, two additional methods of simplifying circuits, are useful when the performance of a circuit focuses on its behavior at a specific pair of terminals. The principle of superposition is an important concept when two or more sources are present in the circuit, because it enables us to isolate the effect of each source on the total response of the circuit.

It is important to keep in mind that although we have used resistive networks to illustrate the various analytical techniques, the techniques themselves are not limited to resistive circuits. In Chapter 7 we begin discussing the analysis of linear, lumped-parameter circuits that contain inductance and capacitance as well as resistance.

In the next chapter we shall show why the node-voltage method can be used to analyze both planar and nonplanar circuits and, in addition, why the mesh-current method can be used to analyze planar circuits. We will also introduce the loop-current method, which can be used to analyze nonplanar circuits.

PROBLEMS

4.1 Use the node-voltage method to find v_1 and v_2 in the circuit shown in Fig. P4.1.

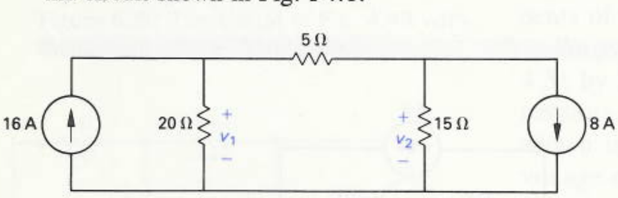


Figure P4.1

4.2 a) Use the node-voltage method to find the branch currents i_1 through i_5 in the circuit shown in Fig. P4.2.

b) Find the total power absorbed by the circuit.

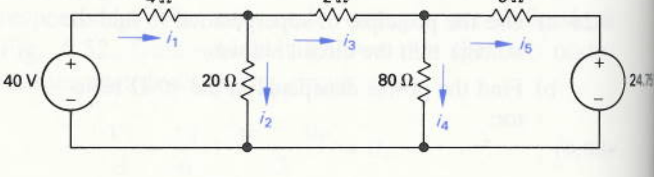


Figure P4.2

4.3 a) Use the node-voltage method to find v_1, v_2, and v_3 in the circuit shown in Fig. P4.3.

b) Check your solution by showing that the total power developed in the circuit equals the total power dissipated.

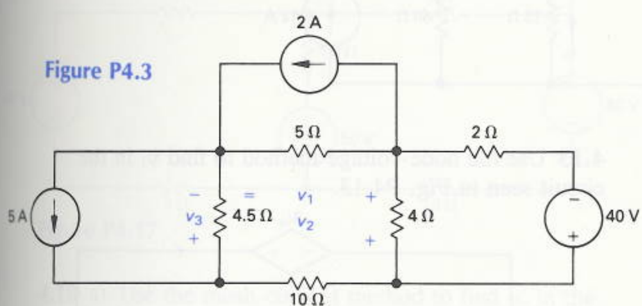


Figure P4.3

4.4 a) Use the node-voltage method to find the power dissipated in each resistor in the circuit shown in Fig. P4.4.

b) Show that the total power dissipated equals the total power delivered.

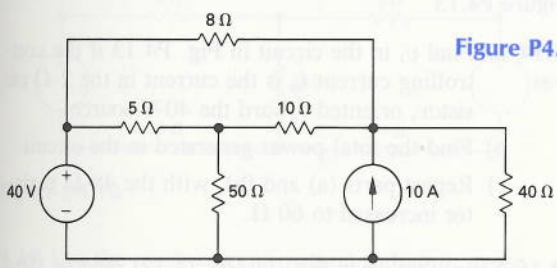


Figure P4.4

4.5 Use the node-voltage method to find the currents i_g and i_o in the circuit shown in Fig. P4.5.

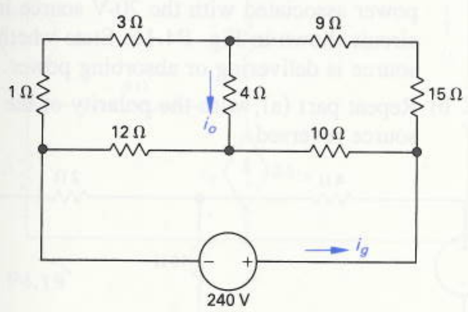


Figure P4.5

4.6 a) Use the node-voltage method to find v_a, v_b, and v_c in the circuit shown in Fig. P4.6.

b) Find the total power absorbed by the circuit.

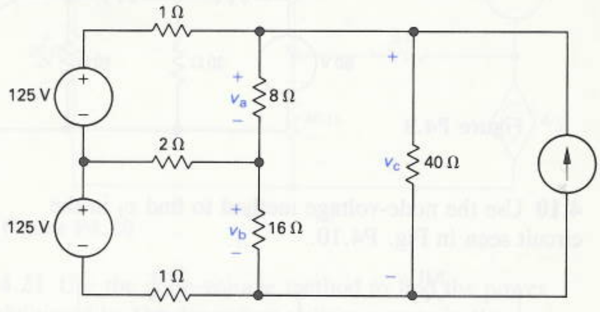


Figure P4.6

4.7 a) Use the node-voltage method to find v_o in the circuit shown in Fig. P4.7.

b) Calculate the power absorbed by the dependent current source.

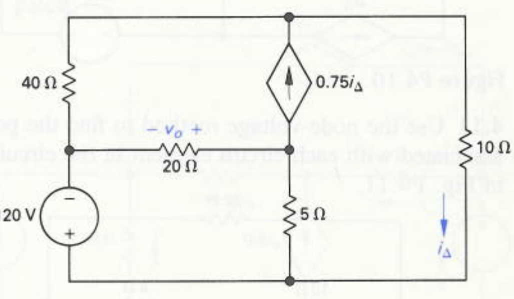


Figure P4.7

4.8 Use the node-voltage method to find v_a in the circuit seen in Fig. P4.8.

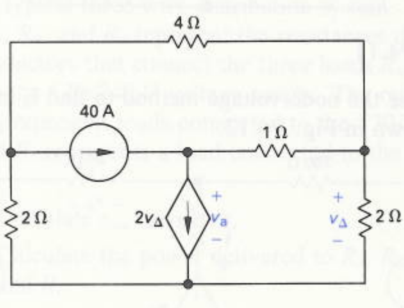


Figure P4.8

4.9 a) Use the node-voltage method to find v_o in the circuit shown in Fig. P4.9.

b) Show that the total power generated equals the total power absorbed.

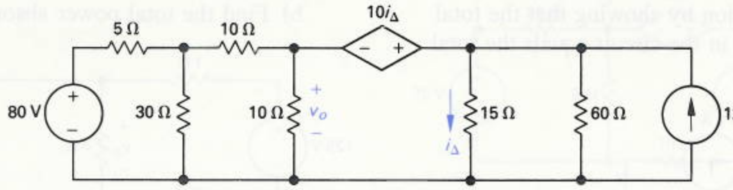


Figure P4.9

4.10 Use the node-voltage method to find v_1 in the circuit seen in Fig. P4.10.

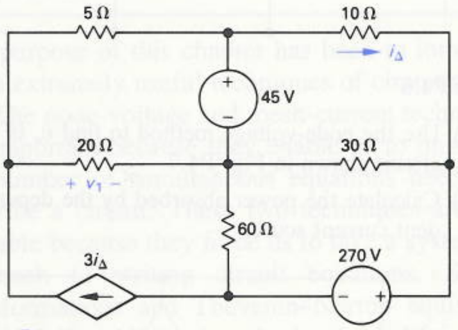


Figure P4.10

4.11 Use the node-voltage method to find the power associated with each circuit element in the circuit seen in Fig. P4.11.

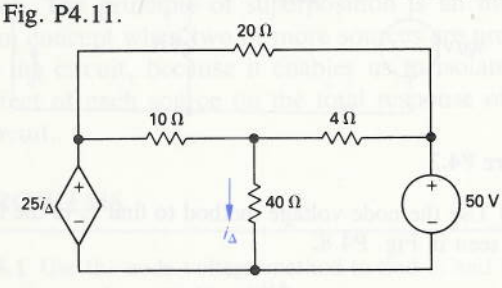


Figure P4.11

4.12 Use the node-voltage method to find i_o in the circuit shown in Fig. P4.12.

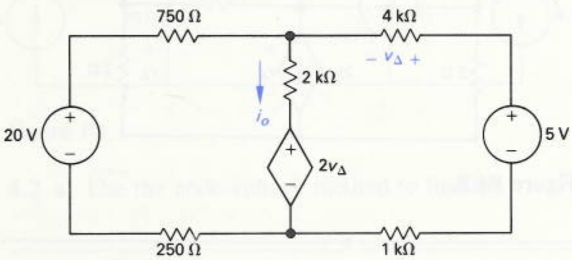


Figure P4.12

4.13 Use the node-voltage method to find v_1 in the circuit seen in Fig. P4.13.

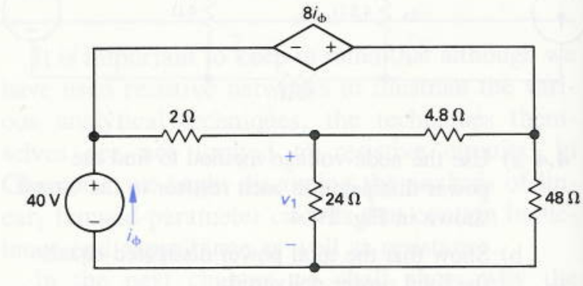


Figure P4.13

4.14 a) Find v_1 in the circuit in Fig. P4.13 if the controlling current i_ϕ is the current in the 2-Ω resistor, oriented toward the 40-V source.

b) Find the total power generated in the circuit.

c) Repeat parts (a) and (b), with the 48-Ω resistor increased to 60 Ω.

4.15 Show that when Eqs. (4.16), (4.17), and (4.19) are solved for i_B, the result is identical to Eq. (2.27).

4.16 a) Use the mesh-current method to find the power associated with the 20-V source in the circuit shown in Fig. P4.16. State whether the source is delivering or absorbing power.

b) Repeat part (a), with the polarity of the 60-V source reversed.

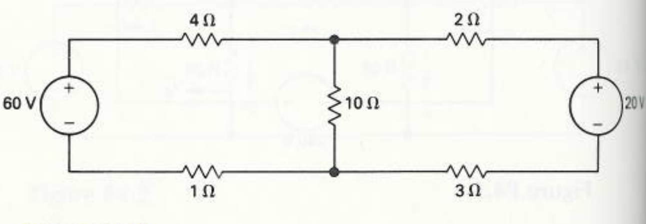


Figure P4.16

4.17 Use the mesh-current method to find the power associated with the 50-V source in the circuit seen in Fig. P4.17. State whether the source is delivering or absorbing power.

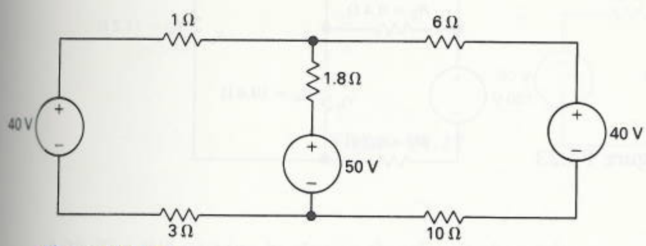


Figure P4.17

4.18 a) Use the mesh-current method to find v_o in the circuit shown in Fig. P4.18.

b) Find the total power dissipated in the circuit.

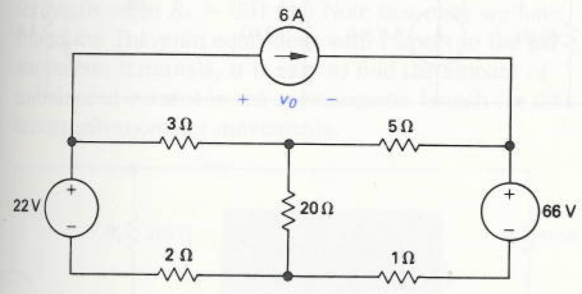


Figure P4.18

4.19 Use the mesh-current method to find v_o in the circuit shown in Fig. P4.19.

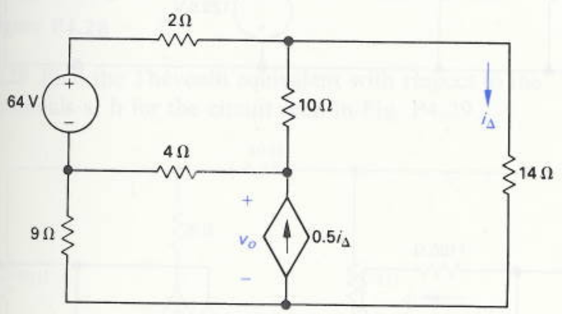


Figure P4.19

4.20 Use the mesh-current method to find the power delivered by the dependent voltage source in the circuit seen in Fig. P4.20.

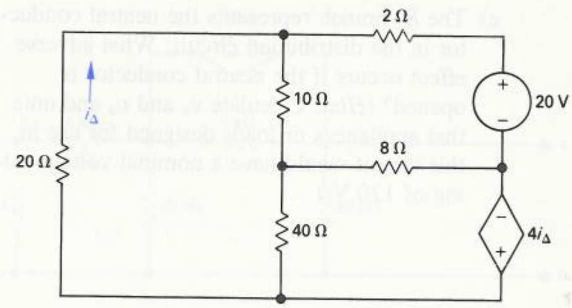


Figure P4.20

4.21 Use the node-voltage method to find the power delivered by the dependent voltage source in the circuit in Fig. P4.20.

4.22 a) Find the branch currents i_a and i_e for the circuit shown in Fig. P4.22.

b) Check your answers by showing that the total power generated equals the total power dissipated.

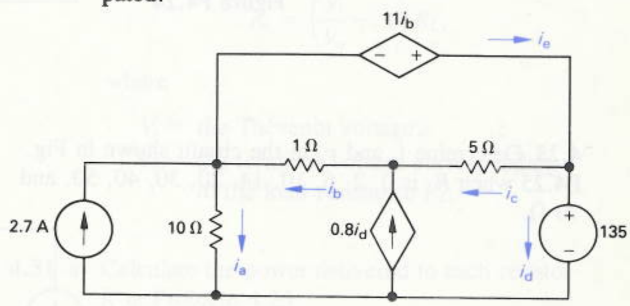


Figure P4.22

4.23 The circuit in Fig. P4.23 is a direct-current version of a typical three-wire distribution system. The resistors R_1, R_2, and R_3 represent the resistances of the three conductors that connect the three loads R_a, R_b, and R_c to the 120/240-V voltage supply. The resistors R_a and R_b represent loads connected to the 120-V circuits, and R_c represents a load connected to the 240-V circuit.

a) Calculate v_a, v_b, and v_c.

b) Calculate the power delivered to R_a, R_b, and R_c.

c) Calculate the power delivered by each source.

d) What percentage of the source power is delivered to the loads?

e) The R_2 branch represents the neutral conductor in the distribution circuit. What adverse effect occurs if the neutral conductor is opened? (*Hint:* Calculate v_a and v_b and note that appliances or loads designed for use in this circuit would have a nominal voltage rating of 120 V.)

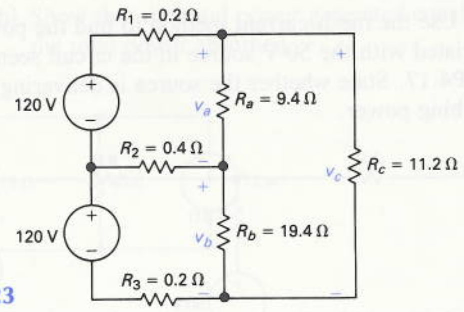


Figure P4.23

4.24 Find the current in the 4-$k\Omega$ resistor in the circuit of Fig. P4.24 by making a succession of appropriate source transformations.

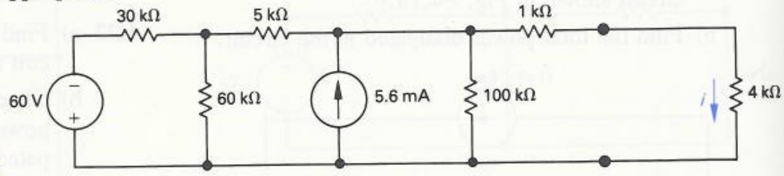


Figure P4.24

4.25 Determine i_o and v_o in the circuit shown in Fig. P4.25 when R_o is 0, 2, 6, 10, 14, 20, 30, 40, 50, and 70 Ω.

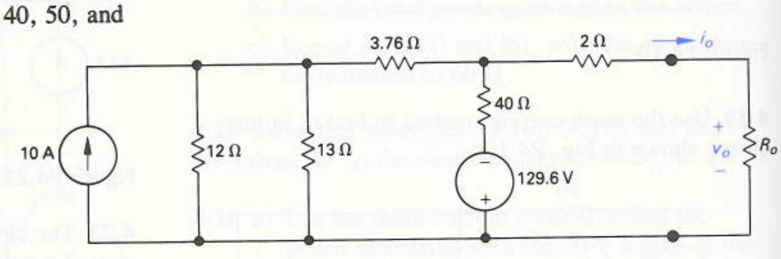


Figure P4.25

4.26 Determine the Thévenin equivalent with respect to the terminals a, b for the circuit shown in Fig. P4.26.

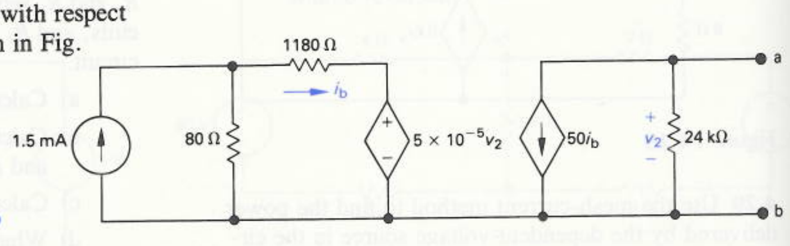


Figure P4.26

4.27 Find the Thévenin equivalent with respect to the terminals a, b for the circuit seen in Fig. P4.27.

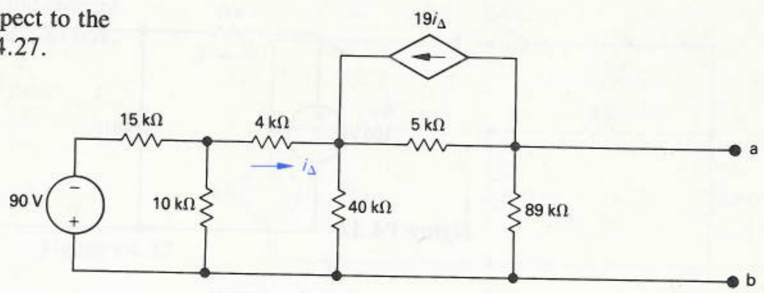


Figure P4.27

4.28 The Wheatstone bridge in the circuit shown in Fig. P4.28 is balanced when R_3 equals 800 Ω. If the galvanometer has a resistance of 50 Ω, how much current will the galvanometer detect when the bridge is unbalanced by setting R_3 to 801 Ω? (*Hint:* Find the Thévenin equivalent with respect to the galvanometer terminals when R_3 = 801 Ω.) Note that once we have found the Thévenin equivalent with respect to the galvanometer terminals, it is easy to find the amount of unbalanced current in the galvanometer branch for different galvanometer movements.

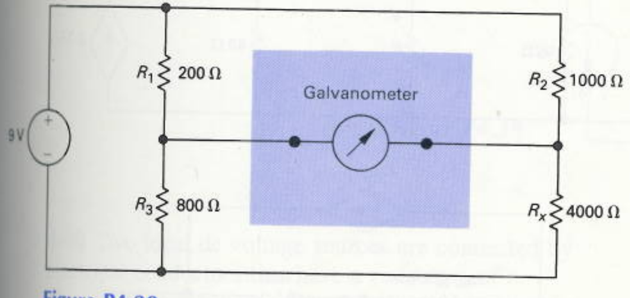


Figure P4.28

4.29 Find the Thévenin equivalent with respect to the terminals a, b for the circuit seen in Fig. P4.29.

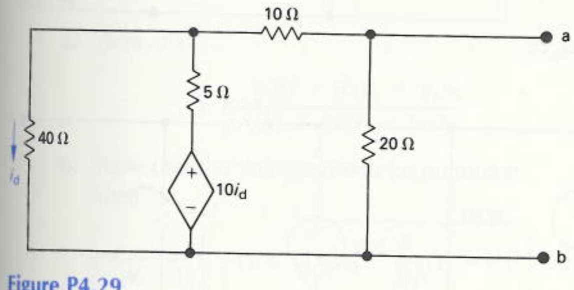


Figure P4.29

4.30 Laboratory measurements on a dc voltage source yield a terminal voltage of 75 V with no load connected to the source and 60 V when loaded with a 20-Ω resistor.

a) What is the Thévenin equivalent with respect to the terminals of the dc voltage source?

b) Show that the Thévenin resistance of the source is given by the expression

$$R_t = \left(\frac{V_t}{V_o} - 1\right) R_L,$$

where

V_t = the Thévenin voltage,

V_o = the terminal voltage corresponding to the load resistance R_L.

4.31 a) Calculate the power delivered to each resistor R in Problem 4.25.

b) Plot the power delivered versus the resistance.

c) At what value of R is the power maximum?

4.32 The resistor R_o in the circuit seen in Fig. P4.32 is adjusted until maximum power is delivered to the resistor.

a) What is the value of R_o?

b) What is the power delivered to R_o?

c) What percentage of the total power developed by the sources in the circuit is delivered to R_o?

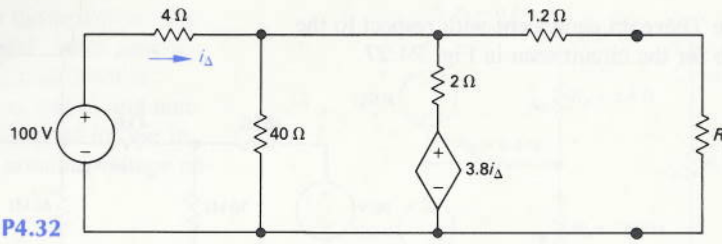


Figure P4.32

4.33 Determine the maximum power that the circuit seen in Fig. P4.33 can deliver to a resistive load connected to the terminals x, y.

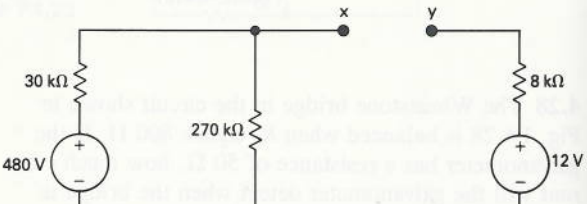


Figure P4.33

4.34 A variable resistor is connected between the terminals a, b in the circuit seen in Fig. P4.34 and adjusted until maximum power is delivered to the resistor. What is the maximum power?

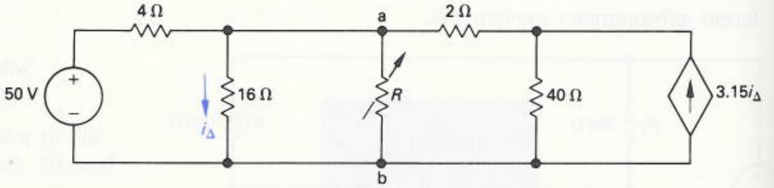


Figure P4.34

4.35 The load resistor R_o in the circuit seen in Fig. P4.35 is adjusted until maximum power is delivered to R_o.

a) What is the value of R_o?

b) What is the maximum power?

c) What percentage of the power developed by the 160-V source is dissipated in R_o?

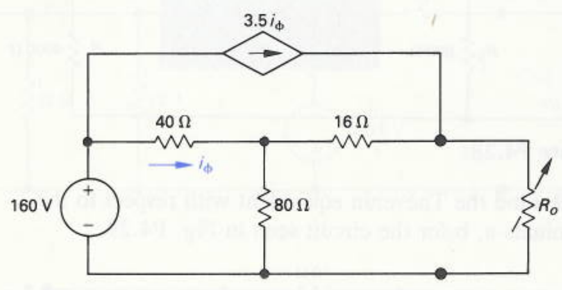


Figure P4.35

4.36 A voltmeter with a resistance of 60 kΩ is used to measure the voltage v_{ab} in the circuit seen in Fig. P4.36.

a) What is the voltmeter reading?

b) What is the percent error in the voltmeter reading if percent error is defined as [(measured-actual)/actual] × 100?

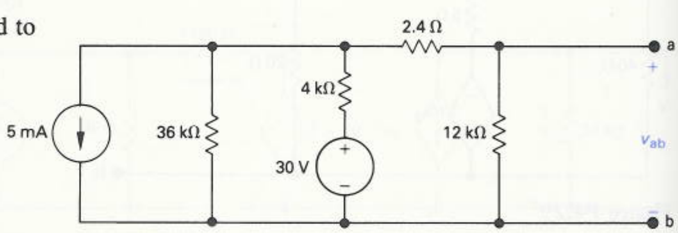


Figure P4.36

4.37 When an ammeter is used to measure the current i_Δ in the circuit shown in Fig. P4.37, it reads 320 mA. What is the resistance of the ammeter?

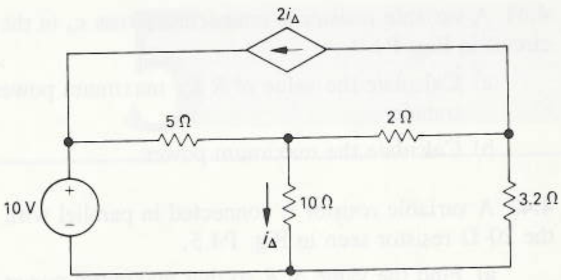


Figure P4.37

4.38 When a voltmeter is used to measure the voltage v_e in Fig. P4.38, it reads 4 V. What is the resistance of the meter?

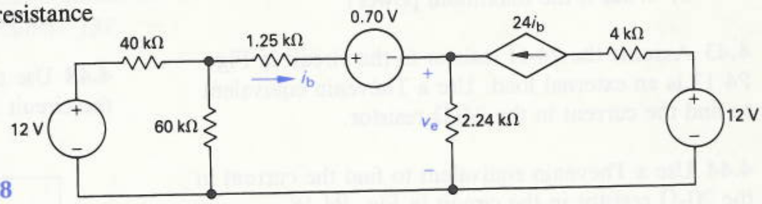


Figure P4.38

4.39 Use the principle of superposition to find the current i_g in the circuit shown in Fig. P4.39.

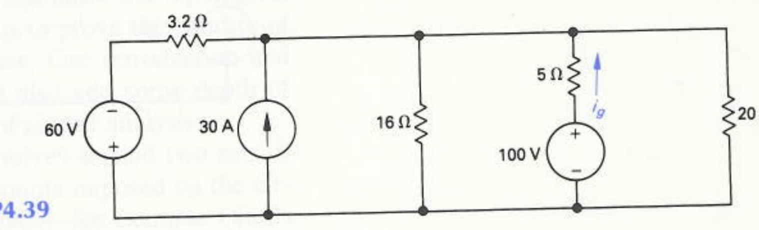


Figure P4.39

4.40 Two ideal dc voltage sources are connected by electrical conductors that have a resistance of r ohms/meter as shown in Fig. P4.40. A load having a resistance of R ohms moves between the two voltage sources. Let x equal the distance between the load and the source V_1 and l equal the distance between the sources.

a) Show that

$$v = \frac{V_1 Rl + R(V_2 - V_1)x}{Rl + 2rlx - 2rx^2}$$

b) Show that the voltage v will be minimum when

$$x = \frac{l}{V_2 - V_1}\left[-V_1 \pm \sqrt{V_1 V_2 - \frac{R}{2rl}(V_1 - V_2)^2}\right].$$

c) Find x when $l = 16$ km, $V_1 = 1000$ V, $V_2 = 1200$ V, $R = 3.9\ \Omega$ and $r = 5 \times 10^{-5}\ \Omega$/m.

d) What is the minimum value of v for the circuit of part (c)?

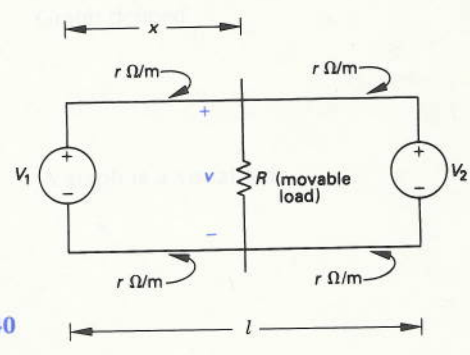


Figure P4.40

4.41 A variable resistor is connected across v_o in the circuit in Fig. P4.9.

a) Calculate the value of R for maximum power transfer.

b) Calculate the maximum power.

4.42 A variable resistor is connected in parallel with the 20-Ω resistor seen in Fig. P4.5.

a) Find the value of R so that maximum power is delivered to R.

b) What is the maximum power?

4.43 Assume the 24-Ω resistor in the circuit in Fig. P4.13 is an external load. Use a Thévenin equivalent to find the current in the 24-Ω resistor.

4.44 Use a Thévenin equivalent to find the current in the 20-Ω resistor in the circuit in Fig. P4.18.

4.45 Use the principle of superposition to find v_o in the circuit in Fig. P4.18.

4.46 Use the principle of superposition to find i_a in the circuit in Fig. P4.22.

4.47 An adjustable resistor R_o is connected in parallel with the 20-Ω resistor seen in the circuit in Fig. P4.10. The resistor is adjusted until maximum power is delivered to its terminals.

a) What is the value of R_o?

b) What is the maximum power delivered to R_o?

4.48 Use the principle of superposition to find v_o in the circuit shown in Fig. P4.48.

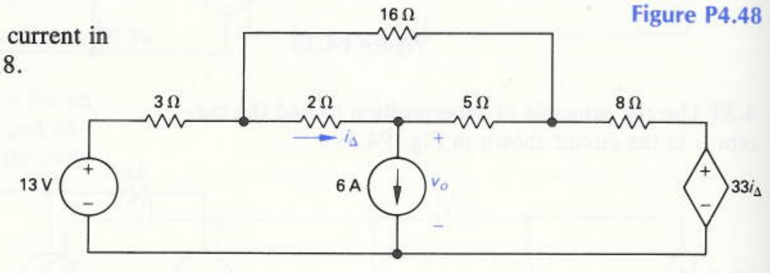


Figure P4.48