6

CIRCUIT ANALYSIS BY LAPLACE TRANSFORMS

OVERVIEW

In Chapter 5 the concept of the Laplace transform was developed, and many of its basic mathematical properties were introduced. Consideration was made both of determining the transform of a given time function and determining the time function corresponding to a given transform function. It should be recalled that these transformations represent the initial and final steps of a complete network solution. There remains the problem of solving the problem in the transform domain.

The major purpose of this chapter is to develop the procedure for solving a complete circuit problem by transform techniques. Transform-domain equivalent circuits are developed for representing the voltage–current relationships of all circuit components. The use of these equivalent circuits permits the application of basic algebraic circuit analysis schemes to be applied directly to complex circuits.

The properties of certain common circuit forms are investigated in some detail. This includes first- and second-order circuits with arbitrary excitations.

OBJECTIVES

After completing this chapter, the reader should be able to

1. Draw the transform equivalent circuit of a *capacitor* and represent the effects of any initial energy storage by a suitable s-domain source.
2. Draw the transform equivalent circuit of an *inductor* and represent the effect of any initial energy storage by a suitable s-domain source.
3. Draw the transform equivalent circuit of a *resistor.*

4. Transform a complete circuit, including the effects of sources, and represent the circuit in the best form for either mesh or node analysis.

5. Apply basic circuit analysis methods (e.g., mesh analysis, node analysis, Thévenin's theorem, etc.) to transform-domain circuits to obtain desired response functions.

6. Discuss the concepts of *natural* and *forced* response and how these terms relate to *transient* and *steady-state* response.

7. Define the terms *impulse response* and *step response*.

8. Determine complete responses of first-order circuits with arbitrary excitations using transform methods.

9. Determine complete responses of second-order circuits with arbitrary excitations using transform methods.

10. For a second-order response, discuss the mathematical and physical nature of the following types of responses: (a) *overdamped*, (b) *critically damped*, and (c) *underdamped*.

11. Define the following terms with respect to a second-order response: (a) *damping constant*, (b) *undamped natural frequency*, and (c) *damped natural frequency*.

12. Analyze the response of a series *RLC* circuit excited by a step function of voltage.

13. Analyze the response of a parallel *RLC* circuit excited by a step function of current.

14. Discuss the relationship between the number of energy-storage elements and the order of the circuit.

15. Solve an ordinary constant-coefficient linear differential equation using transform methods.

16. For a given circuit, write the differential equations directly in the time domain, and solve for a desired variable using transform methods.

6-1 TRANSFORM EQUIVALENT OF CAPACITANCE

In Chapter 3 we studied the voltage–current relationships for a charged capacitor. We found that a charged capacitor could be represented as an uncharged capacitor in series with a dc (or step) voltage source as long as the voltage source was considered to be included in the effective terminals of the capacitor. Since the effect of the initial voltage is treated like other sources in the network, let us consider for the moment an uncharged capacitor as shown in Fig. 6-1. Using $t = 0$ as an initial reference, the

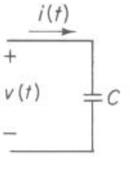

Figure 6-1 Time-domain representation of uncharged capacitor.

time-domain relationship is

$$v(t) = \frac{1}{C}\int_0^t i(t)\,dt \tag{6-1}$$

Now let us "operate" on Eq. (6-1) by determining the Laplace transforms of both sides. Since $v(t)$ and $i(t)$ are not specified, we can use the definitions

$$V(s) = \mathscr{L}[v(t)] \tag{6-2}$$

$$I(s) = \mathscr{L}[i(t)] \tag{6-3}$$

Transformation of Eq. (6-1) yields

$$\mathscr{L}[v(t)] = \mathscr{L}\left[\frac{1}{C}\int_0^t i(t)\,dt\right] \tag{6-4}$$

or

$$V(s) = \frac{1}{C}\mathscr{L}\left[\int_0^t i(t)\,dt\right] \tag{6-5}$$

By means of operation (O-2) of Chapter 5, the Laplace transform of the integral term is

$$\mathscr{L}\left[\int_0^t i(t)\,dt\right] = \frac{I(s)}{s} \tag{6-6}$$

Substitution of Eq. (6-6) into Eq. (6-5) results in

$$V(s) = \frac{1}{sC}I(s) \tag{6-7}$$

A most important characteristic of Eq. (6-7) is that, while the time-domain equation involves an integration, the transform-domain equation is an algebraic equation. Of course, the quantity s appears in the equation, but it may be manipulated by any normal algebraic procedure.

In a dc resistive circuit, the voltage is a constant times the current. Although it is not correct to speak of s as a constant (it is actually a type of *operator*), we can compare Eq. (6-7) to a dc circuit in the sense that the voltage is expressed as a function times the current.

Since we cannot use the term "resistance," we will borrow the term *impedance* from steady-state ac circuit theory and attach the adjective *transform* for descriptive reasons. We define the *transform impedance of a capacitor* as

$$Z(s) = \frac{1}{sC} \tag{6-8}$$

The quantity impedance has the same dimensions as resistance, namely ohms. Impedance in the transform domain may be treated, from an algebraic point of view, in the same manner as resistance is treated in dc circuits. The essential difference is that the s operator is ever present and must not be misplaced in manipulations.

Although we have only looked at a capacitor so far, it will help to slightly generalize the concept at this point. We can define *Ohm's law in the transform domain*

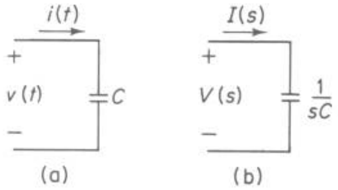

Figure 6-2 Time-domain and transform-domain representations of uncharged capacitor.

by

$$V(s) = Z(s)I(s) \tag{6-9}$$

or

$$I(s) = \frac{V(s)}{Z(s)} \tag{6-10}$$

or

$$Z(s) = \frac{V(s)}{I(s)} \tag{6-11}$$

We can also speak of the reciprocal of impedance. This quantity will be called the *transform admittance* and will be denoted by $Y(s)$. Thus

$$Y(s) = \frac{1}{Z(s)} \tag{6-12}$$

For the capacitor, the transform admittance is

$$Y(s) = sC \tag{6-13}$$

Returning to the capacitor and considering Fig. 6-2a, we can transform the capacitor by expressing it as an impedance $1/sC$ as shown in (b). The circuit is now in a form suitable for transform analysis. Strictly speaking, there is no reason why we could not designate the transformed circuit in terms of admittance. However, for the same reason that resistors are usually designated in ohms instead of siemens, namely consistency, we will normally designate the transforms of circuits in terms of impedances.

6-2 TRANSFORM EQUIVALENT OF INDUCTANCE

In Chapter 3 we found that an inductor with an initial current could be represented as an unfluxed inductor in parallel with a current source as long as the current source was considered to be included in the effective terminals of the inductor. Since the effect of the initial current is treated like other sources in the network, let us consider for the moment an unfluxed inductor as shown in Fig. 6-3. Using $t = 0$ as an initial reference, the time-domain relationship is

$$v(t) = L\frac{di(t)}{dt} \tag{6-14}$$

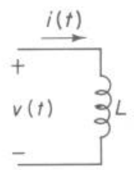

Figure 6-3 Time-domain representation of unfluxed inductor.

Now let us "operate" on Eq. (6-14) by determining the Laplace transforms of both sides. As in the case of the capacitor, and in essentially all future work, we employ the transform definitions.

$$V(s) = \mathscr{L}[v(t)] \tag{6-15}$$

$$I(s) = \mathscr{L}[i(t)] \tag{6-16}$$

Transformation of Eq. (6-14) yields

$$\mathscr{L}[v(t)] = \mathscr{L}\left[L\frac{di}{dt}\right] \tag{6-17}$$

or

$$V(s) = L\mathscr{L}\left[\frac{di}{dt}\right] \tag{6-18}$$

By means of (O-1), with the assumption that the initial current is zero, we obtain

$$\mathscr{L}\left[\frac{di}{dt}\right] = sI(s) \tag{6-19}$$

Substitution of Eq. (6-19) into Eq. (6-18) yields

$$V(s) = sLI(s) \tag{6-20}$$

As in the case of the capacitor, the transform-domain equation for an inductor is an algebraic equation. Thus *the transform impedance of an inductor is*

$$Z(s) = sL \tag{6-21}$$

If the transform admittance of an inductor is desired, it can be expressed as

$$Y(s) = \frac{1}{sL} \tag{6-22}$$

Referring to Fig. 6-4a, we can transform an unfluxed inductor by expressing it as an impedance sL as shown in (b).

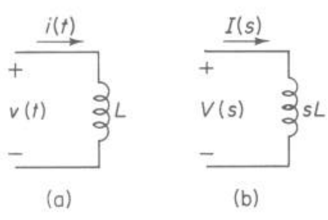

Figure 6-4 Time-domain and transform-domain representations of unfluxed inductor.

6-3 TRANSFORM EQUIVALENT OF RESISTANCE

The transform impedance of a resistor is very easily obtained and needs very little explanation. The time-domain relationship is

$$v(t) = Ri(t) \tag{6-23}$$

Transformation of both sides of Eq. (6-23) yields

$$V(s) = RI(s) \tag{6-24}$$

Thus the *transform impedance of a resistor is simply*

$$Z(s) = R \tag{6-25}$$

6-4 TRANSFORMING COMPLETE CIRCUITS

A complete linear circuit under excitation consists of one or more sources (both voltage and current sources, in general) and any arbitrary combination of circuit components (resistance, capacitance, and inductance, in general). The sources in the network are either actual excitations or hypothetical sources due to initial conditions. *As far as transform manipulations are concerned, initial condition sources are treated exactly like external sources.* The essential difference lies in determining the effective terminals of reactive components.

To transform sources, we apply the techniques developed in Chapter 5 for determining the transforms of time functions. To transform components, we employ the results of the preceding three sections of this chapter.

Example 6-1

The switch in the circuit of Fig. 6-5a is closed at $t = 0$. The initial values of inductive currents and capacitive voltages are shown. Draw the transformed circuit in a form most suitable for mesh current analysis.

Solution Although the circuit may or may not be in a steady-state condition at $t = 0$, so long as we know the initial voltages on capacitors and initial currents through inductors, it is immaterial. We first draw the time-domain circuit, expressing initial condition generators as shown in Fig. 6-5b. Since we are primarily interested in mesh current analysis, we have used the equivalent circuits involving voltage sources. Thus, the two inductor initial condition sources are impulse sources. Recall that in Chapter 4 we stated that, for time-domain differential equation solutions, it was best to avoid impulse functions in general. However, *in transform analysis, impulse sources should be retained in the circuit as they provide the required initial conditions.* Note also that we have converted the right-hand dc current source to a dc voltage source.

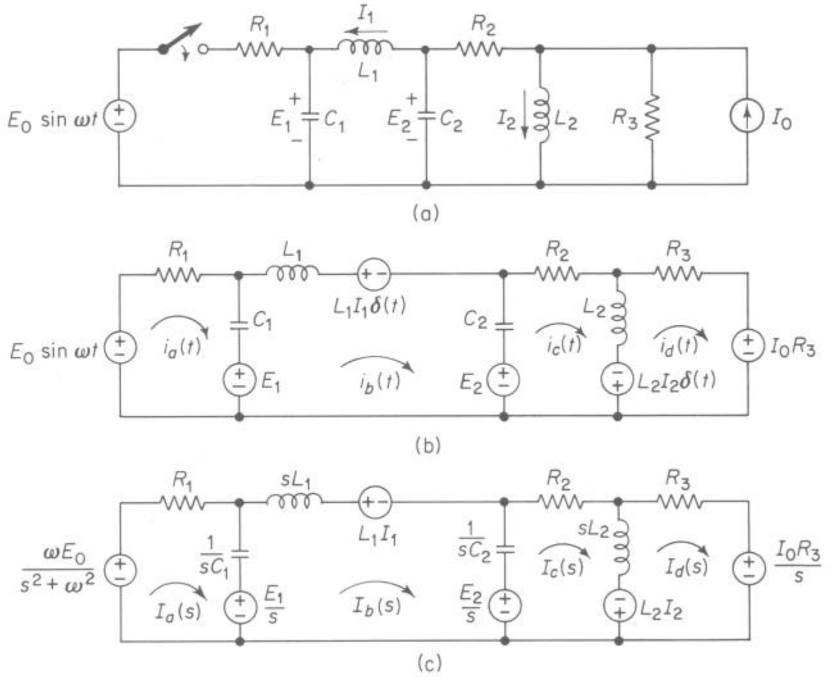

Figure 6-5 Circuit of Ex. 6-1.

Recalling the appropriate transform pairs from Chapter 5, the transformed circuit is shown in Fig. 6-5c. It should be observed again that the transforms of the impulse sources have the simplest possible form, since they are constants.

This circuit could now be solved or simplified by any standard algebraic or circuit analysis procedure for the desired transform variable. Finally, the inverse transform of the quantity desired could be determined. We consider these ideas gradually as we go along; so we will not attempt to solve this circuit at the moment.

6-5 SOLUTIONS OF COMPLETE CIRCUITS IN THE TRANSFORM DOMAIN

After a circuit is completely transformed according to the procedure discussed in Section 6-4, it may then be manipulated by any standard algebraic or circuit analysis technique. Among the possible methods of solution are mesh current analysis, node voltage analysis, Thévenin's theorem, Norton's theorem, successive reduction techniques, and many others. *In general, any dc circuit analysis scheme may be employed as long as we remember that both sources and impedances which appear in the circuit are functions of the variable s.*

After the desired voltage or current is obtained in the transform domain, its inverse transform can be determined to yield the final time function. In many problems such as are encountered in network and control system design studies, the entire analysis and design may be carried out in the transform domain, and no inversion may be necessary. In other words, engineers and technologists have learned to "think in the transform domain." Let us now illustrate the approach of transform solution methods by solving some examples.

Example 6-2

Referring back to the circuit of Example 6-1, write a set of mesh current equations that characterize the network.

Solution The transformed circuit is shown again in Fig. 6-6 with mesh currents assigned. Remember, the transform impedances and transform sources are treated exactly like dc quantities in writing equations. The mesh equations are

$$\left[R_1 + \frac{1}{sC_1}\right] I_a(s) - \left[\frac{1}{sC_1}\right] I_b(s) = \frac{\omega E_0}{s^2 + \omega^2} - \frac{E_1}{s} \qquad (6\text{-}26)$$

$$\left[\frac{-1}{sC_1}\right] I_a(s) + \left[\frac{1}{sC_1} + sL_1 + \frac{1}{sC_2}\right] I_b(s) - \left[\frac{1}{sC_2}\right] I_c(s) = \frac{E_1}{s} - L_1 I_1 - \frac{E_2}{s} \qquad (6\text{-}27)$$

$$-\left[\frac{1}{sC_2}\right] I_b(s) + \left[\frac{1}{sC_2} + R_2 + sL_2\right] I_c(s) - [sL_2] I_d(s) = \frac{E_2}{s} + L_2 I_2 \qquad (6\text{-}28)$$

$$-[sL_2] I_c(s) + [sL_2 + R_3] I_d(s) = -L_2 I_2 - \frac{I_0 R_3}{s} \qquad (6\text{-}29)$$

The reader should be careful not to be confused by the use of capital letters for both the transform quantities and initial conditions, since the quantities are quite different. We have carefully kept the argument (s) to ensure that the transform functions are recognized. The reader is urged to do the same when working problems.

Theoretically, the equations could be solved simultaneously to yield any desired current, and of course, branch voltages and currents could be determined from the

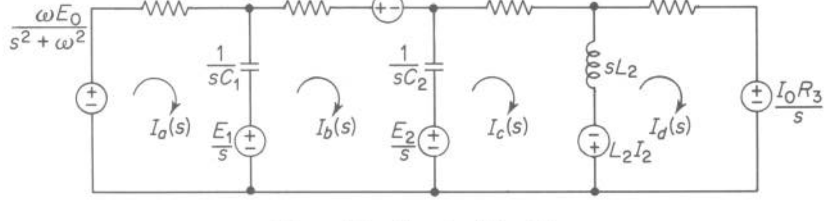

Figure 6-6 Circuit of Ex. 6-2.

mesh currents. Practically speaking, however, this could be quite a chore from a computational viewpoint for this circuit. Also, to obtain the inverse transform of a particular current would be most unwieldy even if values were specified for the components. The completion of a problem as complex as this would be enhanced by the availability of computer facilities. Thus we will leave this problem without further consideration.

Example 6-3

The switch in Fig. 6-7a is opened at $t = 0$, thus applying the current source to the network. Write the transform node voltage equations, determine $V_2(s)$, and invert to determine $v_2(t)$. The capacitor is initially uncharged.

Solution The transformed circuit applicable for $t > 0$ is shown in Fig. 6-7b. Although in node voltage analysis we will work with *admittances*, we have still followed the consistent convention of labeling the schematic in terms of impedances. The equations are

$$V_1(s)[2 + 8s] - V_2(s)[8s] = \frac{10}{s} \tag{6-30}$$

$$-V_1(s)[8s] + V_2(s)[4 + 8s] = 0 \tag{6-31}$$

These equations must be solved simultaneously for $V_2(s)$. This may be achieved by either determinant methods or by substitution. The result is

$$V_2(s) = \frac{10}{6s + 1} = \frac{5/3}{s + 1/6} \tag{6-32}$$

Inversion of $V_2(s)$ follows directly from (T-4). Thus

$$v_2(t) = \frac{5}{3}\epsilon^{-t/6}$$

This circuit could have easily been solved by means of the single time constant circuit concept of Chapter 4. We have solved it by transform methods for illustrative purposes.

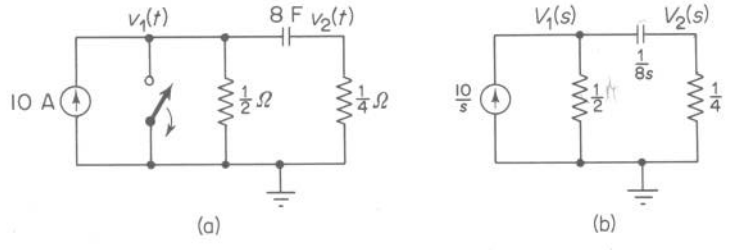

Figure 6-7 Circuit of Ex. 6-3.

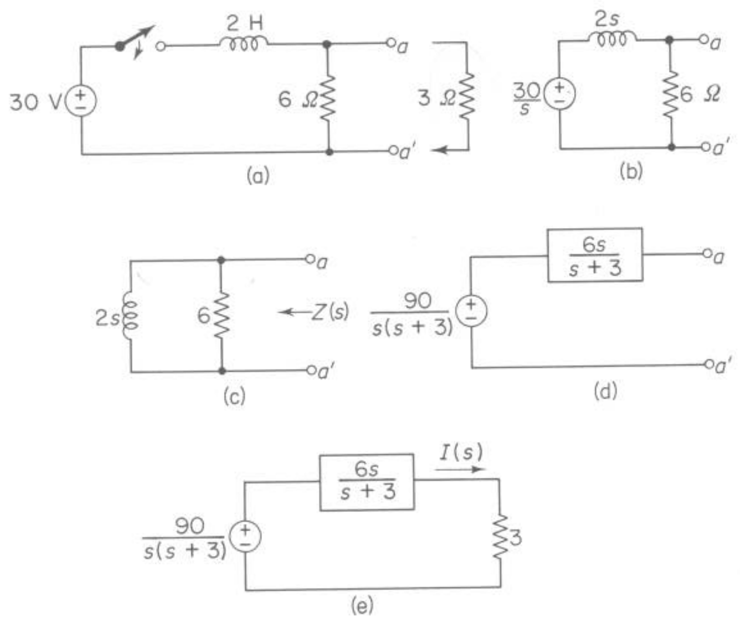

Figure 6-8 Circuit of Ex. 6-4.

Example 6-4

The switch in the circuit of Fig. 6-8a is closed at $t = 0$. The inductor is initially unfluxed.

(a) Determine a transform Thévenin equivalent circuit at the terminals a-a'.

(b) Using the result of (a), determine the current through a 3-Ω res stor which is assumed connected to the circuit at the time the switch is closed.

Solution (a) The transformed circuit is shown in Fig. 6-8b. To obtain the Thévenin equivalent circuit, we first measure $V_{oc}(s)$ across the terminals a-a'. By means of the voltage-divider rule, this voltage is

$$V_{oc}(s) = \frac{30}{s} \times \frac{6}{2s + 6} = \frac{90}{s(s + 3)} \tag{6-33}$$

We determine the Thévenin equivalent impedance $Z(s)$ by shorting the source and combining the impedances, $3s$ and 6, in parallel. Thus, as shown in (c),

$$Z(s) = \frac{2s \times 6}{2s + 6} = \frac{6s}{s + 3} \tag{6-34}$$

The Thévenin equivalent circuit is shown in (d).

(b) Knowing the Thévenin equivalent circuit, we can determine the current through the 3-Ω resistor from the series circuit shown in Fig. 6-8e. We have

$$I(s) = \frac{V_{oc}(s)}{Z(s) + 3} = \frac{90/s(s+3)}{(6s)/(s+3) + 3} = \frac{10}{s(s+1)} \tag{6-35}$$

We may readily expand $I(s)$ into partial fractions and obtain

$$I(s) = \frac{10}{s} - \frac{10}{s+1} \tag{6-36}$$

The time function is

$$i(t) = 10(1 - \epsilon^{-t}) \tag{6-37}$$

Example 6-5

Consider the circuit of Fig. 6-9a, which is identical to the circuit of Example 6-4, except that the circuit to the left of a-a' has been connected for a sufficiently long time so that a steady-state situation exists at $t = 0^-$. The 3-Ω resistor is to be connected at $t = 0$.

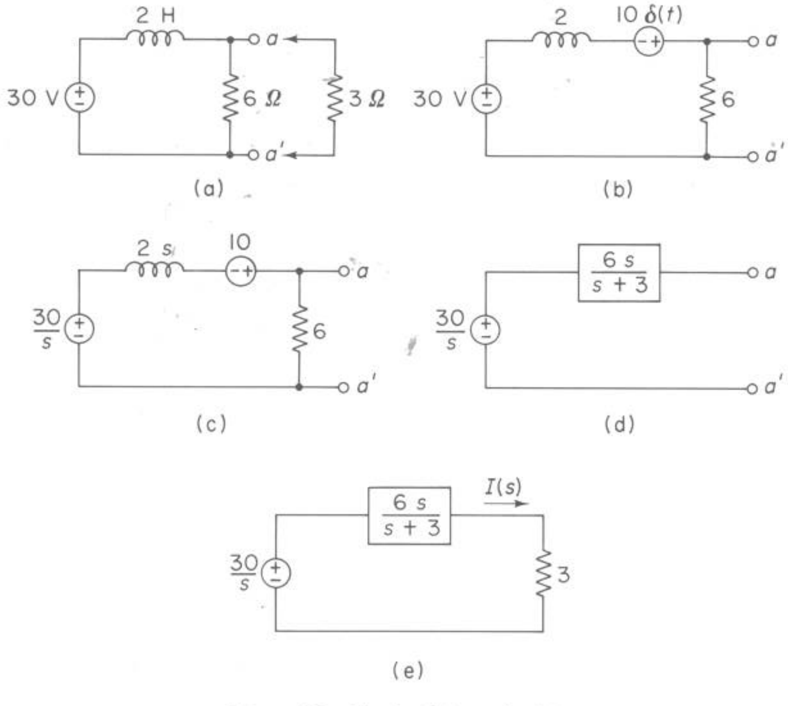

Figure 6-9 Circuit of Example 6-5.

(a) Determine the transform Thévenin equivalent circuit applicable for $t > 0$.

(b) Determine the current through the 3-Ω resistor.

Solution (a) First, we must determine the initial current through the inductor. Since the inductor acts as a short circuit at $t = 0^-$, we have

$$i_L(0^-) = \frac{30}{6} = 5 \text{ A} \tag{6-38}$$

Replacing the fluxed inductor by its Thévenin equivalent circuit, we obtain the circuit shown in Fig. 6-9b. The transformed circuit is shown in (c). The open-circuit voltage in this case is

$$V_{oc}(s) = \left[\frac{30}{s} + 10\right] \times \frac{6}{2s+6} = \frac{30}{s} \tag{6-39}$$

The transform impedance is the same as in Example 6-4, namely,

$$Z(s) = \frac{6s}{s+3} \tag{6-40}$$

The equivalent circuit is shown in (d). Notice that the presence of the initial current through the inductor modifies the Thévenin voltage.

(b) With the 3-Ω resistor connected as shown in Fig. 6-9e, the current is

$$I(s) = \frac{30/s}{(6s)/(s+3) + 3} = \frac{(10/3)(s+3)}{s(s+1)} \tag{6-41a}$$

Partial fraction expansion of $I(s)$ yields

$$I(s) = \frac{10}{s} - \frac{20/3}{s+1} \tag{6-41b}$$

The time function is

$$i(t) = 10 - \frac{20}{3}\epsilon^{-t} \tag{6-42}$$

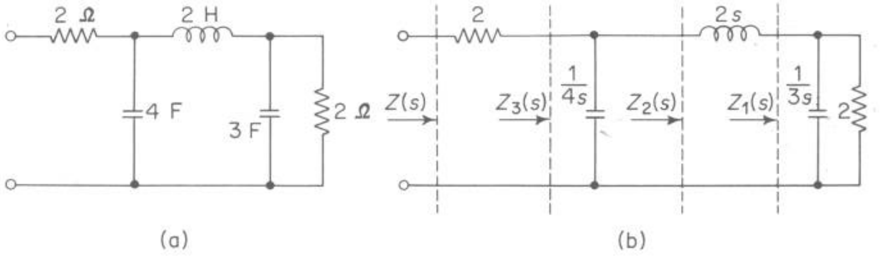

Figure 6-10 Circuit of Ex. 6-6.

Example 6-6

(a) Determine the equivalent transform impedance at the terminals of the relaxed network shown in Fig. 6-10a.

(b) Determine the transform of the current that would flow from a sinusoidal voltage generator, $e(t) = 10 \sin 4t$, connected to the terminals.

Solution (a) Due to the ladder configuration of this circuit, it is best solved by means of successive series and parallel reductions. First we draw the transformed circuit as shown in Fig. 6-10b. The various Z's refer to the impedances looking to the right at various points *if* the circuits to the left, in each case, were disconnected. We have

$$Z_1(s) = \frac{2 \times (1/3s)}{2 + (1/3s)} = \frac{2}{6s+1} \tag{6-43}$$

$$Z_2(s) = 2s + \frac{2}{6s+1} = \frac{12s^2 + 2s + 2}{6s+1} \tag{6-44}$$

At this point in determining $Z_3(s)$, it is probably best to switch to admittance momentarily. Thus

$$Y_3(s) = 4s + \frac{6s+1}{12s^2 + 2s + 2} = \frac{48s^3 + 8s^2 + 14s + 1}{12s^2 + 2s + 2} \tag{6-45}$$

or

$$Z_3(s) = \frac{12s^2 + 2s + 2}{48s^3 + 8s^2 + 14s + 1} \tag{6-46}$$

and

$$\begin{aligned} Z(s) &= 2 + \frac{12s^2 + 2s + 2}{48s^3 + 8s^2 + 14s + 1} \\ &= \frac{96s^3 + 28s^2 + 30s + 4}{48s^3 + 8s^2 + 14s + 1} \end{aligned} \tag{6-47}$$

(b) The sinusoidal excitation applied to the network is given by

$$e(t) = 10 \sin 4t \tag{6-48}$$

The transform is

$$E(s) = \frac{40}{s^2 + 16} \tag{6-49}$$

The transform current is

$$I(s) = \frac{E(s)}{Z(s)} = \frac{40(48s^3 + 8s^2 + 14s + 1)}{(s^2 + 16)(96s^3 + 28s^2 + 30s + 4)} \tag{6-50}$$

6-6 FORMS FOR INITIAL CONDITIONS

The initial conditions that must be considered in transform analysis are initial currents through inductors and initial voltages on capacitors. In the past few sections, we have dealt with such conditions by first drawing the equivalent time-domain circuits and then transforming the hypothetical sources along with other sources in the network. This approach is certainly correct and adequate. However, the reader who may have occasion to work many problems involving initial conditions may wish to avoid the intermediate step of writing down the time-domain equivalent circuit. Instead, it is possible, with a little practice, to go directly to the transform representations for initial conditions. Also, it is easier to manipulate between Thévenin and Norton forms in the transform domain.

At the beginning, we recognize that since only step and impulse sources are involved, the transforms will be either of the form $1/s$ or a constant. First, let us consider a charged capacitor as shown in Fig. 6-11a. If we mentally visualize replacing the capacitor by its Thévenin equivalent circuit and transforming, we obtain the representation shown in (b). Thus, the transform Thévenin equivalent circuit follows naturally from basic considerations. To obtain the Norton equivalent circuit shown in (c), we apply Norton's theorem directly in the transform domain. The short-circuit current would be

$$I_{sc}(s) = \frac{V_0/s}{1/sC} = CV_0 \tag{6-51}$$

and the impedance is simply $1/sC$. Thus, the Norton equivalent circuit is easily obtained directly in the transform domain, without dealing specifically with an impulse function. If desired, one can begin each problem with the easily remembered Thévenin form.

The opposite situation exists in the case of the fluxed inductor shown in Fig. 6-12a. The most natural representation is the Norton equivalent circuit whose transform is shown in (b). To obtain the Thévenin equivalent circuit shown in (c), we could measure V_{oc} and would obtain

$$V_{oc}(s) = \frac{I_0}{s}\, sL = LI_0 \tag{6-52}$$

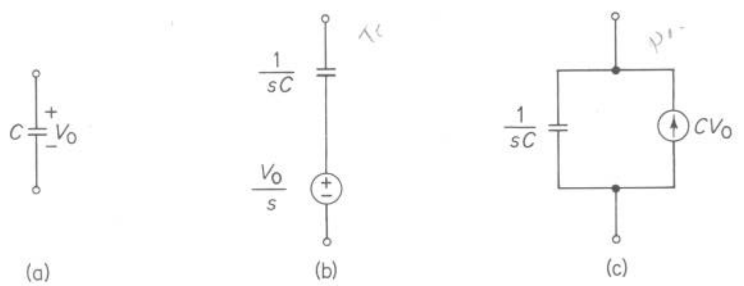

Figure 6-11 Transform representations for charged capacitor.

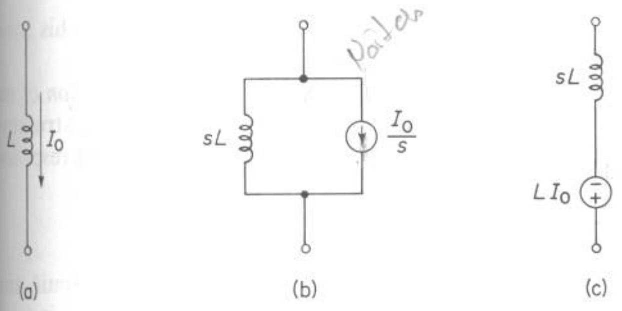

Figure 6-12 Transform representation for fluxed inductor.

and the impedance is sL. Thus, in the case of the inductor, the easily remembered form is the Norton form, and the Thévenin form may be easily obtained in the transform domain. As in the time domain, the sources must be considered to be within the effective terminals of the component.

6-7 TRANSIENT AND STEADY-STATE PHENOMENA

The concepts of transient and steady-state phenomena were introduced in Chapter 4. It was stated then that, immediately after a circuit is excited by one or more arbitrary excitations, there is an initial "adjustment" response called the *transient response.* Eventually, in most cases, the circuit reaches an equilibrium or semiequilibrium state in which the response follows a more uniform behavior. The latter type of response is called the *steady-state response.*

Before we pursue any more transform problems, it is wise to pause here and look at this concept again in view of having solved a few transform problems. Such considerations will help us to "see the forest through the trees" as we go along.

From the problems already considered, we note that the nature of the time function is dependent on the type of poles (denominator roots) of the desired transform variable. Corresponding to each pole (or pair of poles for complex poles), there is a corresponding term in the time response. If a given pole is real and simple, the time response is an exponential term. If a pair of poles are complex, the time response is a damped sinusoidal term, etc. Thus *a knowledge of the poles of a transform response is sufficient to predict the type of time response.* Of course, inverse transformation is necessary to determine the actual magnitudes of the various terms, but a quick inspection to determine the type of response is often helpful.

As we work through many problems in the remainder of this chapter, the reader should observe that the poles of a given transform response arise from two distinct sources: (a) poles due to the network structure, and (b) poles due to the excitation. Thus, any time response, in general, will consist of (a) terms due to the circuit itself, and (b) terms due to the excitation.

The poles due to the network structure may be considered to produce the portion of the total response known as the natural response. In essentially all practical circuits not containing electronic feedback or negative resistance, the natural response vanishes after a sufficiently long time. In such cases, this natural response may be considered to be the *transient response*, although the remainder of the total response is simultaneously present and affects the entire shape of the response during this transient phase.

The poles due to the excitation may be considered to produce the portion of the total response known as the forced response. If the excitation and network structure are such that the natural response eventually vanishes, the remaining forced response may be considered to be the *steady-state response.*

In summary:

1. The form of the *natural response* is determined by the nature of the circuit, and this function may be considered to be the *transient response* whenever it is transient in nature.
2. The form of the *forced response* is determined by the nature of the excitation, and this function may be considered to be the *steady-state* response whenever the natural response is transient in nature.

In studying the natural transient behavior of circuits, it is often necessary to assume some specific type of excitation. The two most common types of excitations for this purpose are the impulse function and the step function. The impulse function is particularly simple for this purpose since $\mathscr{L}[\delta(t)] = 1$. Thus since the transform of the impulse function contains no poles at all, the forms of all terms of the response resulting from such an excitation are due to the network alone and are thus *natural response terms.*

Although the transform of the step function is not quite as simple since it contains a pole, $s = 0$, in some cases the response resulting from a step excitation is simpler in form than the response resulting from an impulse excitation. This is particularly true in cases where the pole of the transform of the step function cancels a numerator factor, resulting only in poles due to the network structure. Furthermore, the response resulting from a step excitation may be readily measured in the laboratory.

A response resulting from an impulse excitation is called an *impulse response*, and a response resulting from a step excitation is called a *step response.* Both the impulse response and step response concepts are widely used in systems analysis.

6-8 FIRST-ORDER CIRCUITS

In this section and in the next few sections, we consider some specific classes of circuits that are readily solvable by transform methods. The purpose is twofold: (a) we can certainly gain more practice in transform analysis by solving more problems, and (b) we can learn to associate certain types of responses with certain special classes of networks that occur frequently in practical applications.

In this section we consider a few problems involving circuits whose time-domain differential equations are of first order. Such circuits were considered in Chapter 4 and were appropriately designated as single time constant circuits. It was demonstrated that such circuits consisted of either resistance and capacitance (RC) or resistance and inductance (RL). Such circuits have appeared in a few examples in this chapter.

When single time constant circuits were considered in the time domain in Chapter 4, we restricted the excitations to include only dc sources and initial conditions. For those cases it was indicated that the solutions could be obtained most easily by inspection. However, there is certainly no reason why such circuits couldn't be solved by transform methods. When the general framework of analysis is in the transform domain, it may be more convenient to do so. Furthermore, and most important, with transform methods, we can easily generalize our excitations to include such waveforms as impulse functions, sinusoidal functions, and many others.

An important point to remember is that with arbitrary excitations, the responses may involve more than just exponential terms and a dc term even though the circuit itself is of the single time constant form. Such general terms are due to the nature of the excitation terms. Thus if a sinusoidal source excites a single time constant circuit, the response may consist of an exponential term and a sinusoidal term. We have preferred to define a different term for describing such circuits with general excitations. We will refer to a circuit that can be described by a first-order differential equation with arbitrary excitations as a *first-order circuit*. Let us consider some examples.

Example 6-7

The circuit shown in Fig. 6-13a is excited at $t = 0$ by a voltage source $e(t)$. The capacitor is initially uncharged. By transform methods, determine the current $i(t)$ and the voltage across the capacitor $v(t)$, for $t > 0$, for each of the following excitations.

(a) $e(t) = 20$ V (dc source)
(b) $e(t) = 20 \sin 2t$
(c) $e(t) = 20\epsilon^{-t}$
(d) $e(t) = 20\epsilon^{-2t}$

Solution First, we transform the circuit as shown in Fig. 6-13b. We have used a general $E(s)$ at this point since we ultimately must consider four different cases. Since two variables, $i(t)$ and $v(t)$, must be determined in each case, we could solve for one of the desired variables in the transform domain, invert it, and determine the other variable quite easily in the time domain, since the voltage and current associated with a capacitor are easily related. However, to enhance our understanding of transform manipulations, we will choose to

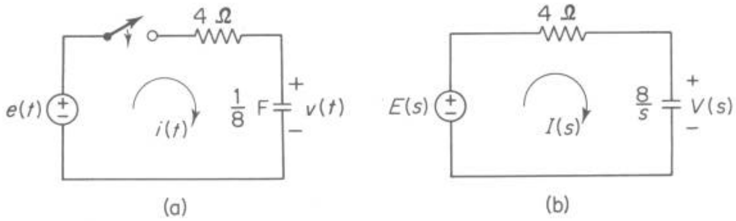

Figure 6-13 Circuit of Ex. 6-7.

solve for all variables in the transform domain and invert each separate quantity, except for the case of excitation (b).

First, let us obtain a general expression for $I(s)$ and $V(s)$ in terms of $E(s)$ so that we may "plug in" the different values of $E(s)$. From the circuit of Fig. 6-13b, we have

$$I(s) = \frac{E(s)}{4 + (8/s)} = \frac{sE(s)}{4(s + 2)} \tag{6-53}$$

The quantity $V(s)$ is given by

$$V(s) = \frac{8}{s} I(s) = \frac{2E(s)}{s + 2} \tag{6-54}$$

Now let us consider each of the excitations. The reader is encouraged to check the case (a) by use of the single time constant concept of Chapter 4.

(a) $e(t) = 20\ V$. The transform of $e(t)$ is

$$E(s) = \frac{20}{s} \tag{6-55}$$

Substitution of $E(s)$ into Eq. (6-53) yields

$$I(s) = \frac{5}{s + 2} \tag{6-56}$$

Thus, by means of pair (T-4), we have

$$i(t) = 5\epsilon^{-2t} \tag{6-57}$$

Substitution of $E(s)$ into Eq. (6-54) yields for $V(s)$

$$V(s) = \frac{40}{s(s + 2)} = \frac{20}{s} - \frac{20}{s + 2} \tag{6-58}$$

Thus

$$v(t) = 20(1 - \epsilon^{-2t}) \tag{6-59}$$

Sketches of $i(t)$ and $v(t)$ are shown in Fig. 6-14a. It is observed that the steady-state response of $i(t)$ is zero, whereas the steady-state response of $v(t)$ is a constant 20 V.

(b) $e(t) = 20 \sin 2t$.

$$E(s) = \frac{40}{s^2 + 4} \tag{6-60}$$

$$I(s) = \frac{10s}{(s + 2)(s^2 + 4)} \tag{6-61}$$

We may readily determine the portion of the response due to the real pole of simple order by partial-fraction expansion. Calling this portion of the

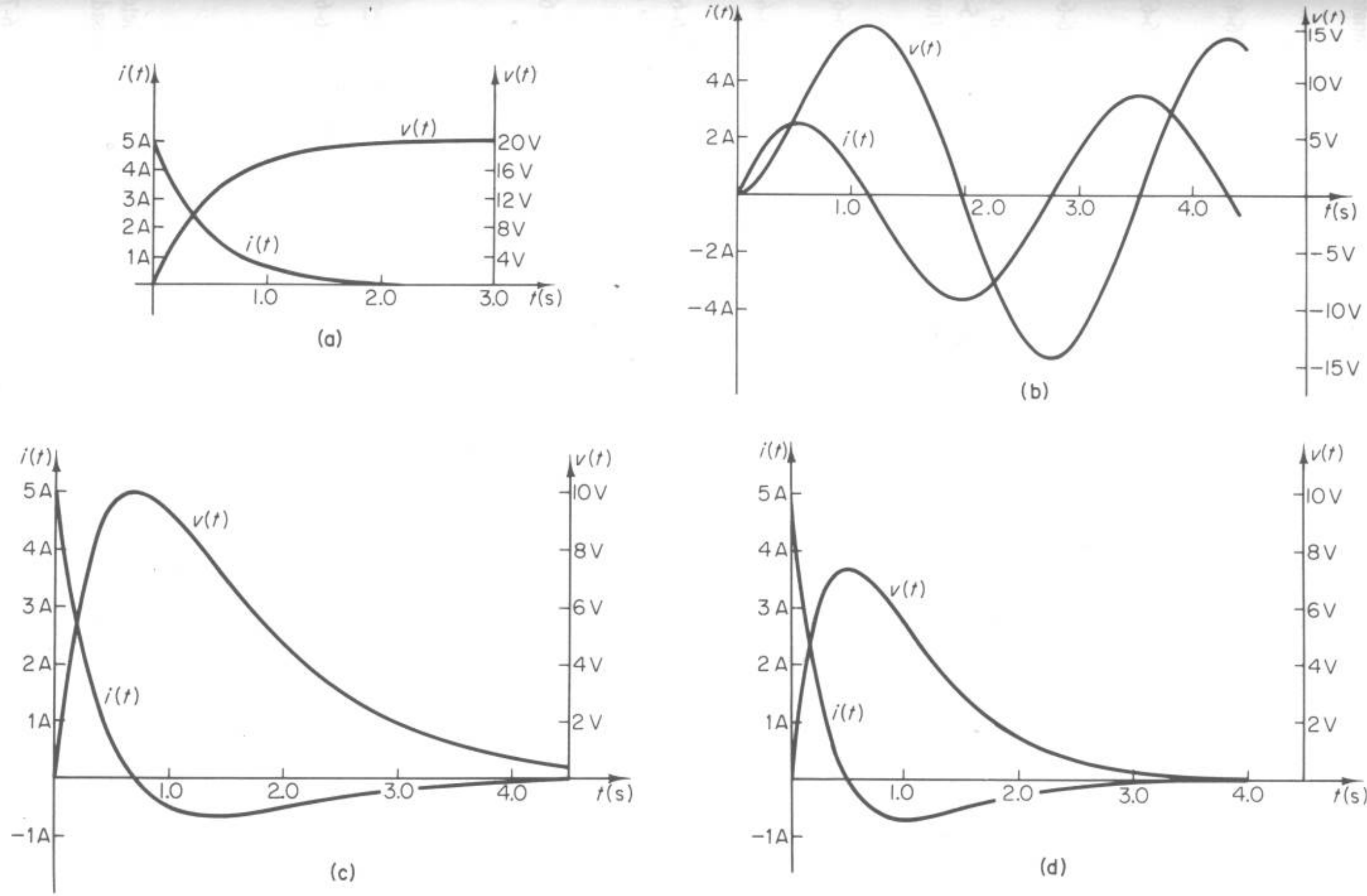

Figure 6-14 Responses of circuit of Ex. 6-7.

expression $I_1(s)$, we have

$$I_1(s) = \frac{A}{s+2} \tag{6-62}$$

$$A = \left.\frac{10s}{s^2+4}\right]_{s=-2} = -\frac{5}{2} \tag{6-63}$$

$$i_1(t) = -\frac{5}{2}\epsilon^{-2t} \tag{6-64}$$

The sinusoidal component of the time response, which is due to the pair of imaginary roots, can be determined by the "trick" formula of Section 5-7. Letting $i_2(t)$ represent this function and employing the notation of that section, we have

$$Q(s) = \frac{10s}{s+2} \tag{6-65}$$

$$Q(j2) = \frac{20j}{2+j2} = \frac{10}{\sqrt{2}}\,\underline{/45^\circ} \tag{6-66}$$

$$i_2(t) = \frac{5}{\sqrt{2}}\sin(2t+45^\circ) \tag{6-67}$$

Thus,

$$i(t) = i_1(t) + i_2(t) = -\frac{5}{2}\epsilon^{-2t} + \frac{5}{\sqrt{2}}\sin(2t+45^\circ) \tag{6-68}$$

In this case, we will determine $v(t)$ in the time domain

$$\begin{aligned} v(t) &= 8\int_0^t i(t)\,dt \\ &= 8\int_0^t \left[-\frac{5}{2}\epsilon^{-2t} + \frac{5}{\sqrt{2}}\sin(2t+45^\circ)\right]dt \\ &= \left[10\epsilon^{-2t} - \frac{20}{\sqrt{2}}\cos(2t+45^\circ)\right]_0^t \\ &= 10[\epsilon^{-2t} - \sqrt{2}\cos(2t+45^\circ)] \end{aligned} \tag{6-69}$$

Sketches of $i(t)$ and $v(t)$ are shown in Fig. 6-14b.

The transient and steady-state portions of the responses are readily determined. After the exponential term becomes negligible, the remaining steady-state response in each case is a sinusoidal function. Thus

$$i_{ss}(t) = \frac{5}{\sqrt{2}}\sin(2t+45^\circ) \tag{6-70}$$

and

$$v_{ss}(t) = -10\sqrt{2}\cos(2t+45^\circ) \tag{6-71}$$

(c) $e(t) = 20\epsilon^{-t}$.

$$E(s) = \frac{20}{s+1} \tag{6-72}$$

$$I(s) = \frac{5s}{(s+1)(s+2)} = \frac{-5}{s+1} + \frac{10}{s+2} \tag{6-73}$$

$$i(t) = 10\epsilon^{-2t} - 5\epsilon^{-t} = 5\epsilon^{-t}(2\epsilon^{-t} - 1) \tag{6-74}$$

$$V(s) = \frac{40}{(s+1)(s+2)} = \frac{40}{s+1} - \frac{40}{s+2} \tag{6-75}$$

$$v(t) = 40(\epsilon^{-t} - \epsilon^{-2t}) = 40\epsilon^{-t}(1 - \epsilon^{-t}) \tag{6-76}$$

Sketches of $i(t)$ and $v(t)$ are shown in Fig. 6-14c.

In this case, the entire responses are transient in nature, and the steady-state solution is zero as a result of the nature of the excitation.

(d) $e(t) = 20\epsilon^{-2t}$. The reader may wonder why we wish to consider another exponential function since, at first glance, this might appear to be a repeat of the type of solution in (c). However, further calculation will reveal the difference. In this case,

$$E(s) = \frac{20}{s+2} \tag{6-77}$$

Substitution of $E(s)$ into Eq. (6-53) yields

$$I(s) = \frac{5s}{(s+2)^2} \tag{6-78}$$

We note that in this case the time constant of the excitation coincides with the time constant of the circuit, thus producing poles of second order in the response. This is a special case of forced resonance which will be discussed more fully later. Referring to the procedure of Section 5-8, we have

$$I(s) = \frac{A_1}{(s+2)^2} + \frac{A_2}{(s+2)} \tag{6-79}$$

$$Q(s) = 5s \tag{6-80}$$

$$A_1 = 5s]_{s=-2} = -10 \tag{6-81}$$

$$A_2 = 5 \tag{6-82}$$

Thus

$$i(t) = -10t\epsilon^{-2t} + 5\epsilon^{-2t} = 5\epsilon^{-2t}(1 - 2t) \tag{6-83}$$

Substituting $E(s)$ into Eq. (6-54) we have for $V(s)$

$$V(s) = \frac{40}{(s+2)^2} \tag{6-84}$$

In this case the expression is recognized to be of the form of (T-10). Thus

$$v(t) = 40t\epsilon^{-2t} \tag{6-85}$$

Sketches of $i(t)$ and $v(t)$ are shown in Fig. 6-14d. As in the case of (c), the entire responses are transient in nature.

Example 6-8

Consider the circuit of Example 6-7, which is shown again in Fig. 6-15a. However, now the capacitor is initially charged to 5 V in the direction shown. The circuit is excited at $t = 0$ by the exponential source shown. Solve for $i(t)$ and $v(t)$.

Solution First, we transform the circuit as shown in Fig. 6-15b. The current in this case is given by

$$I(s) = \frac{[20/(s+1)] + (5/s)}{4 + (8/s)} = \frac{5s}{(s+1)(s+2)} + \frac{1.25}{(s+2)} \tag{6-86}$$

Although we could readily expand both terms in partial fraction expansions, if we refer back to part (c) of Example 6-7, we note that the first term is identical with the entire expression for $I(s)$ obtained in that case. Thus the second term in this case must be due to the initial voltage on the capacitor. The reader is invited to use the superposition principle on this circuit and verify that the second term is, in fact, equal to the current that would be produced by the 5-V source acting alone.

In any event, a complete partial fraction expansion yields

$$I(s) = \frac{-5}{s+1} + \frac{11.25}{s+2} \tag{6-87}$$

Thus

$$i(t) = 11.25\epsilon^{-2t} - 5\epsilon^{-t} \tag{6-88}$$

Again, we will choose to find $V(s)$ and invert. We have from the transform circuit

$$\begin{aligned} V(s) &= \frac{8}{s} I(s) - \frac{5}{s} \\ &= \frac{40}{(s+1)(s+2)} + \frac{10}{s(s+2)} - \frac{5}{s} \end{aligned} \tag{6-89}$$

A partial fraction expansion reads

$$V(s) = \frac{40}{s+1} - \frac{45}{s+2} \tag{6-90}$$

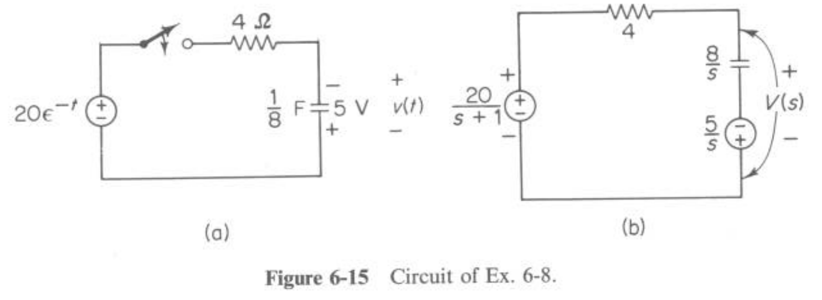

Figure 6-15 Circuit of Ex. 6-8.

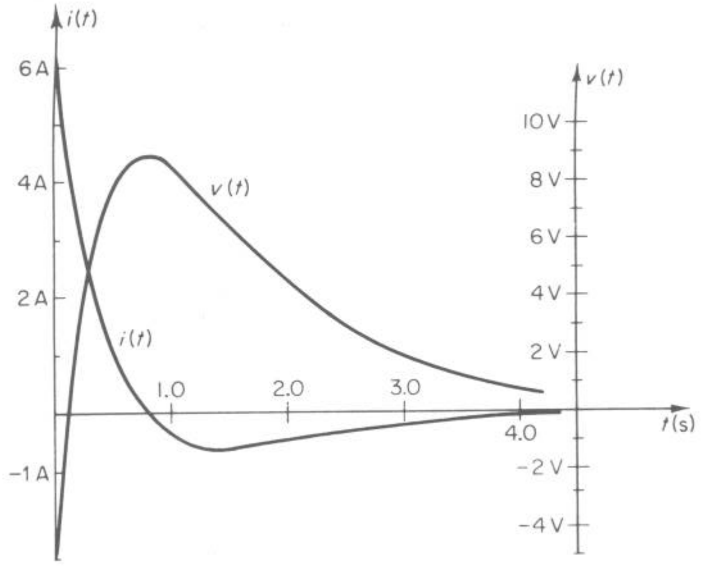

Figure 6-16 Responses of circuit of Ex. 6-8.

in which the net coefficient of the $1/s$ term turns out to be zero. Inversion yields

$$\begin{aligned} v(t) &= 40\epsilon^{-t} - 45\epsilon^{-2t} \\ &= 5\epsilon^{-t}(8 - 9\epsilon^{-t}) \end{aligned} \quad (6\text{-}91)$$

Sketches of $i(t)$ and $v(t)$ are shown in Fig. 6-16.

Example 6-9

The RL circuit of Fig. 6-17a is excited at $t = 0$ by a narrow voltage pulse whose magnitude is 1000 V and whose width is 10 ms as shown in (b).

(a) Using transform analysis, solve for the exact current response $i(t)$.

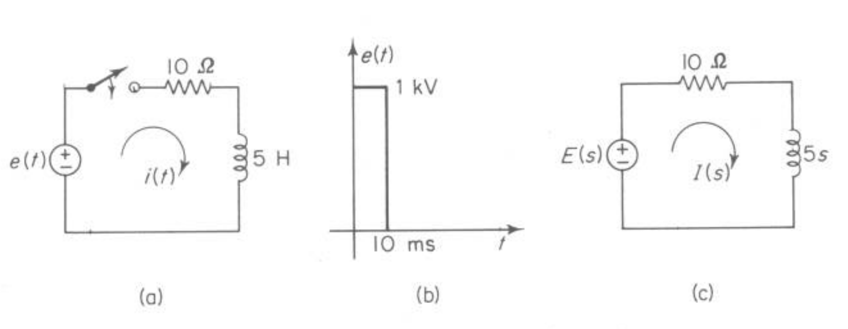

Figure 6-17 Circuit and excitation of Ex. 6-9.

(b) Justify using the impulse approximation and solve for the current response due to an impulse approximation of the excitation. Compare the two results.

Solution (a) The transformed circuit is shown in Fig. 6-17c. To obtain $E(s)$, we first write $e(t)$ as the sum of two step functions.

$$\begin{aligned} e(t) &= 1000u(t) - 1000u(t - 0.01) \\ &= 1000[u(t) - u(t - 0.01)] \end{aligned} \tag{6-92}$$

Using pair (T-2) and operation (O-3), we have

$$E(s) = \frac{1000}{s}[1 - \epsilon^{-0.01s}] \tag{6-93}$$

In terms of $E(s)$, the current is

$$I(s) = \frac{E(s)}{5s + 10} = \frac{200}{s(s + 2)}[1 - \epsilon^{-0.01s}] \tag{6-94}$$

The time response is recognized as being composed of a set of terms beginning at $t = 0$ and a set of terms beginning at $t = 0.01$ s.

Let us designate the first set of terms as $i_0(t)$ and the second set of terms as $i_1(t)$. Referring to their transforms, we have

$$I_0(s) = \frac{200}{s(s + 2)} = \frac{100}{s} - \frac{100}{s + 2} \tag{6-95}$$

and

$$i_0(t) = 100(1 - \epsilon^{-2t})u(t) \tag{6-96}$$

where the $u(t)$ has been used for clarity. Next we have

$$I_1(s) = \frac{-200\epsilon^{-0.01s}}{s(s + 1)} \tag{6-97}$$

To expand $I_1(s)$, we mentally remove the delay term until after the expansion has been carried out, and then place it as a factor for all terms. In other words, it should *not* be kept in the expression when we are determining the partial fraction coefficients, as this may lead to an erroneous interpretation. Thus

$$I_1(s) = -\frac{200}{s(s + 1)}\epsilon^{-0.01s} = -\left[\frac{100}{s} - \frac{100}{s + 2}\right]\epsilon^{-0.01s} \tag{6-98}$$

By means of (O-3) used in conjunction with the appropriate pairs, we have

$$i_1(t) = -100[1 - \epsilon^{-2(t-0.01)}]u(t - 0.01) \tag{6-99}$$

Finally,

$$\begin{aligned} i(t) &= i_0(t) + i_1(t) \\ &= 100(1 - \epsilon^{-2t})u(t) - 100[1 - \epsilon^{-2(t-0.01)}]u(t - 0.01) \end{aligned} \tag{6-100}$$

If desired, we may write $i(t)$ in two different intervals as follows:

$0 < t < 0.01$ s:

$$i(t) = 100(1 - \epsilon^{-2t}) \tag{6-101}$$

$t > 0.01$ s:

$$\begin{aligned} i(t) &= 100 - 100\epsilon^{-2t} - 100 + 100\epsilon^{-2t}\epsilon^{0.02} \\ &= 100\epsilon^{-2t}(\epsilon^{0.02} - 1) \approx 2\epsilon^{-2t} \end{aligned} \tag{6-102}$$

since

$$\epsilon^{0.02} - 1 \approx 0.02 \tag{6-103}$$

A sketch of $i(t)$ is shown in Fig. 6-18.

(b) Let us now investigate the possibility of approximating the narrow pulse as an impulse excitation. In previous situations of this sort, we have assumed that the width of pulses used were short in comparison to the circuit time constants. In this case, we observe that the time constant of the circuit is 0.5 s and that the pulse width is 0.01 s. Thus the time constant is 50 times the width of the pulse, and we may proceed to use the approximation. The area of the pulse is

$$\text{area} = 1000 \text{ V} \times 0.01 \text{ s} = 10 \text{ V}\cdot\text{s} \tag{6-104}$$

Let us designate the impulse approximation to the source as $e_a(t)$ and the response to this impulse as $i_a(t)$. We have

$$e_a(t) = 10\delta(t) \tag{6-105}$$

$$E_a(s) = 10 \tag{6-106}$$

Referring to the circuit, we have

$$I_a(s) = \frac{10}{5s + 10} = \frac{2}{s + 2} \tag{6-107}$$

and

$$i_a(t) = 2\epsilon^{-2t} \tag{6-108}$$

The impulse response is shown on the same scale as the actual response in Fig. 6-18. From the figure and the results of part (a) of this problem, it is seen that the results coincide almost exactly for $t > 0.01$ s, whereas for $t < 0.01$ s, the two responses differ. However, we recognize that the time scale relative to

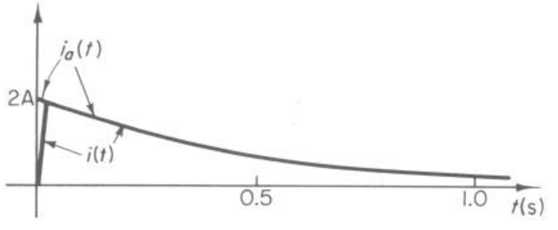

Figure 6-18 Responses of Ex. 6-9.

the time constant of the circuit would probably be the scale of interest in this problem, and hence, the deviation in the initial interval may not be important at all. The savings in computation by using the impulse approximation is obvious. Thus, along with other uses of the impulse function, it may be used as approximation to a very narrow pulse. From the point of view of reasonable accuracy, the shortest time constant of a circuit should be at least 10 to 20 times longer than the pulse width.

6-9 SERIES RLC CIRCUIT

In Section 6-8 our consideration was directed toward first-order circuits, in which case the configuration could be an *RL* or an *RC* form, but could never be a *RLC* or an *LC* form. In this section we begin considering second-order circuits, that is, circuits whose describing differential equation is of second order.

A second-order system may still be an *RL* or an *RC* form, or it may be an *RLC* form. Specifically in this section, we introduce the second-order circuit by means of a special case of importance, the series *RLC* circuit.

Consider the circuit shown in Fig. 6-19a, with no initial energy storage assumed, and its transform shown in (b). Depending on the desired quantity, we can solve for a transform response by writing a mesh current equation or by means of the impedance concept. The latter interpretation results in

$$I(s) = \frac{E(s)}{Z(s)} \tag{6-109}$$

where

$$\begin{aligned} Z(s) &= sL + R + \frac{1}{sC} \\ &= \frac{s^2LC + sRC + 1}{sC} \\ &= \frac{s^2 + sR/L + 1/LC}{s/L} \end{aligned} \tag{6-110}$$

Substitution of Eq. (6-110) into Eq. (6-109) yields

$$I(s) = \frac{sE(s)/L}{s^2 + sR/L + 1/LC} \tag{6-111}$$

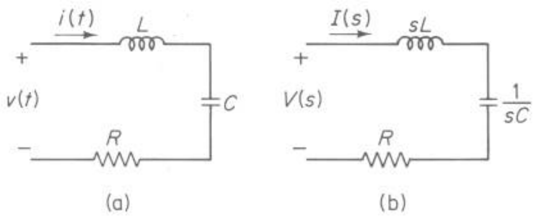

Figure 6-19 Series *RLC* circuit and its transform.

The poles of $I(s)$, which determine the form of the time response, are determined by the poles of $E(s)$ and the roots of the quadratic $s^2 + sR/L + 1/RC$. The latter roots are the zeros (numerator roots) of the impedance $Z(s)$. Since $E(s)$ may be almost anything in general, let us turn our attention at the moment to the form of the transient response of the network due primarily to the poles of the circuit. In this case we assume a step function excitation because of its practical significance and because the simplest possible expression for $I(s)$ will result.

Therefore, let us assume that $e(t)$ is a dc voltage of E volts applied at $t = 0$. The transform of the excitation is $E(s) = E/s$. Substituting $E(s)$ into Eq. (6-111) results in

$$I(s) = \frac{E/L}{s^2 + s(R/L) + 1/LC} \tag{6-112}$$

Notice that the pole, $s = 0$, of $E(s)$ canceled the s term in the numerator of Eq. (6-111). The poles due to the network are determined from the equation

$$s^2 + s\frac{R}{L} + \frac{1}{LC} = 0 \tag{6-113}$$

Letting s_1 and s_2 represent these poles, we have

$$\begin{Bmatrix} s_1 \\ s_2 \end{Bmatrix} = \frac{-R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \tag{6-114}$$

We need to consider three separate cases:

1. *Overdamped.* If $R/2L > 1/\sqrt{LC}$, the roots are both real, negative in sign, and of simple order. The circuit is said to be *overdamped.* In this case, we may write

$$I(s) = \frac{E/L}{(s + \alpha_1)(s + \alpha_2)} = \frac{A_1}{s + \alpha_1} + \frac{A_2}{s + \alpha_2} \tag{6-115}$$

where $\alpha_1 = -s_1$ and $\alpha_2 = -s_2$. The coefficients A_1 and A_2 can be readily determined by partial fraction expansion. The time response is then of the form

$$i(t) = A_1\epsilon^{-\alpha_1 t} + A_2\epsilon^{-\alpha_2 t} \tag{6-116}$$

Thus, the overdamped second-order system produces two separate time constants in the time response. A typical response is shown in Fig. 6-20a.

2. *Critically damped.* If $R/2L = 1/\sqrt{LC}$, the roots are both real, negative in sign, and equal. The circuit is said to be *critically damped.* In this case, we may write

$$I(s) = \frac{E/L}{(s + \alpha)^2} \tag{6-117}$$

where $\alpha = -R/2L$. By means of (T-10), we determine the time response to be

$$i(t) = \frac{Et}{L}\epsilon^{-\alpha t} \tag{6-118}$$

A typical response is shown in Fig. 6-20b.

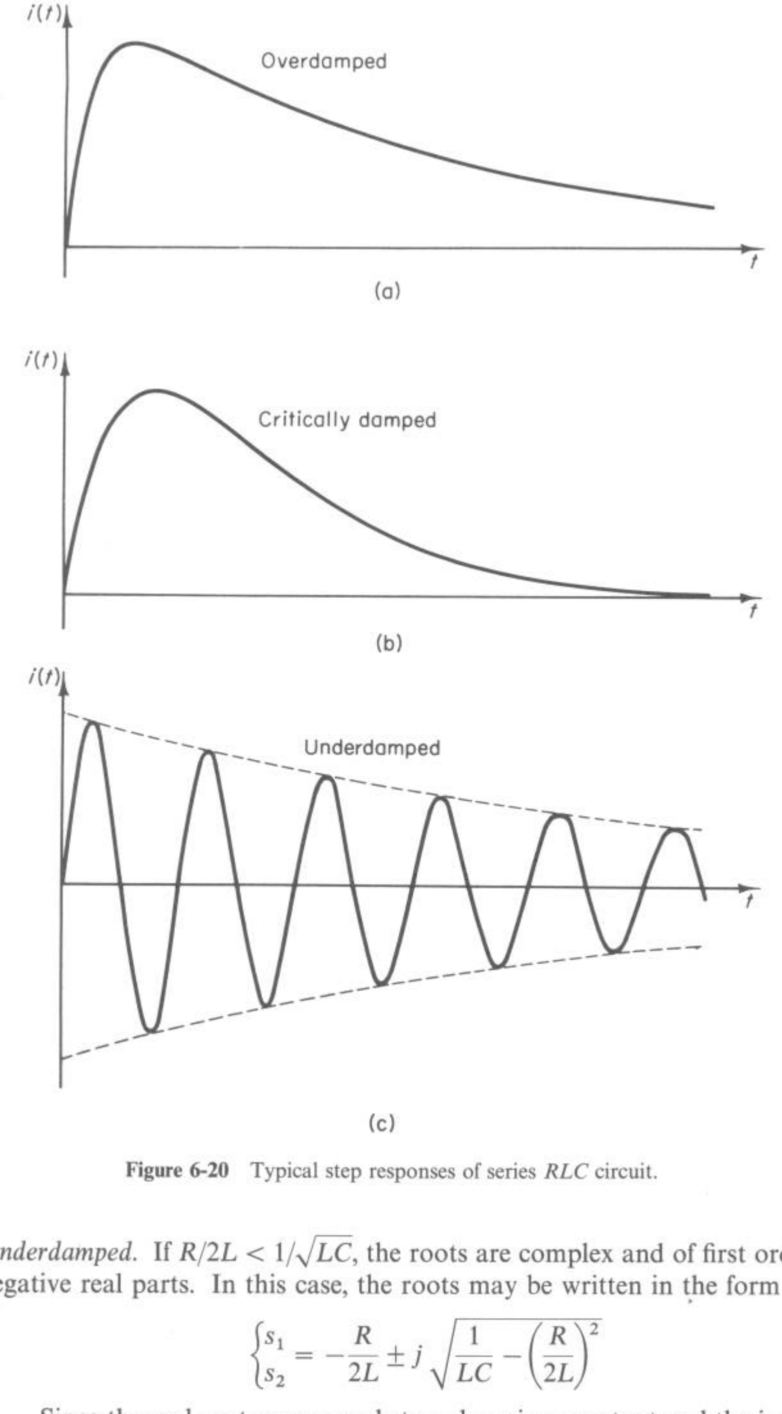

Figure 6-20 Typical step responses of series *RLC* circuit.

3. *Underdamped.* If $R/2L < 1/\sqrt{LC}$, the roots are complex and of first order with negative real parts. In this case, the roots may be written in the form

$$\begin{cases} s_1 \\ s_2 \end{cases} = -\frac{R}{2L} \pm j\sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2} \tag{6-119}$$

Since the real part corresponds to a damping constant and the imaginary part corresponds to an oscillatory response, let us define some useful terms. Let

$$\alpha = \frac{R}{2L} = \text{damping constant} \tag{6-120}$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \text{undamped natural resonant frequency} \tag{6-121}$$

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} = \text{damped natural resonant frequency} \tag{6-122}$$

The quantity ω_0 is the angular frequency of oscillation if there were no resistance in the circuit (i.e., $R = 0$). However, the damped frequency ω_d is always less than the undamped frequency, as can be seen from Eq. (6-122).

For the underdamped case, we may write

$$I(s) = \frac{E/L}{(s + \alpha)^2 + \omega_d^2} \tag{6-123}$$

By means of (T-7), the time response is readily determined to be

$$i(t) = \frac{E\epsilon^{-\alpha t}}{\omega_d L} \sin \omega_d t \tag{6-124}$$

A typical response is shown in Fig. 6-20c.

It is an interesting problem to investigate how the response changes as the damping factor is increased. Consider the circuit of Fig. 6-21 in which L and C are fixed but R is adjustable. Thus ω_0 is fixed, but α varies directly with R and ω_d decreases with an increase of R, according to Eq. (6-122). The results are shown in Fig. 6-22 for some different ratios of α to ω_0. The circuit becomes overdamped for $\alpha/\omega_0 = \zeta = 1$.

When an *RLC* circuit is excited by a more general excitation, the response will consist of two parts. The *natural* part will be due to the circuit itself and will always be similar to one of the forms discussed in this section, depending on whether the circuit is underdamped, critically damped, or overdamped. As long as there is any resistance at all in the circuit, this response will be transient in nature and will disappear after a sufficiently long time, as can be seen from the previous figures. The *forced* part of the response will be due to the nature of the source, and if the source is such as to maintain a response after the transient disappears, such response is, of course, the steady-state response as previously discussed.

In this section we have considered the behavior of *natural resonance* in a typical *RLC* circuit. The reader should not confuse this concept with the closely related concept of *forced resonance* in the sinusoidal steady-state analysis of ac circuits. In

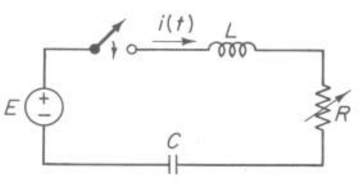

Figure 6-21 Circuit in which the damping is to varied.

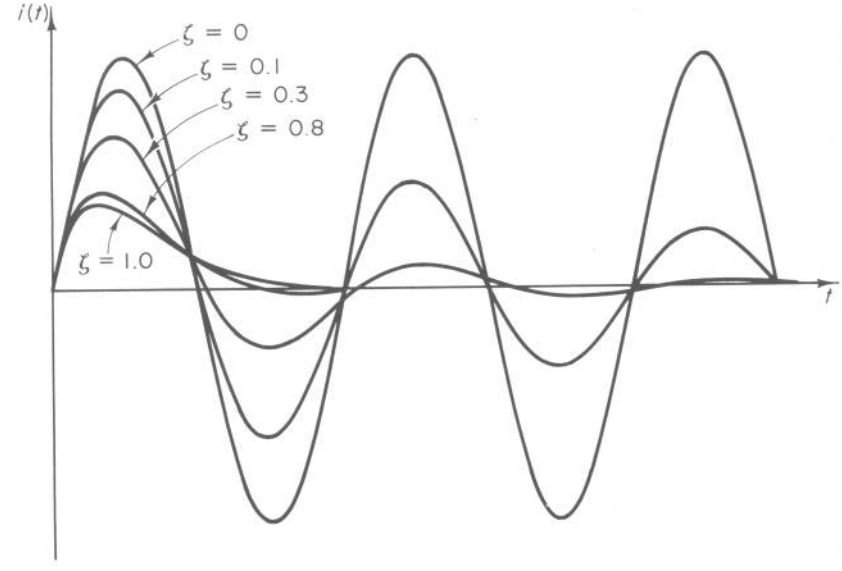

Figure 6-22 Step responses of series *RLC* circuit for different values of damping.

natural resonance, we deal with an oscillation whose form is produced by the circuit itself without regard to the type of excitation. In forced resonance, we deal with circuits having specific steady-state impedance properties, such as minimum or maximum impedances, unity power factor, etc. In fact, the forced steady-state resonant angular frequency of a series *RLC* circuit is usually defined by Eq. (6-121) irrespective of the amount of resistance present. At this frequency, the steady-state reactances cancel, and the impedance is minimum. However, the natural resonance concepts under consideration have a different meaning. The adjectives *natural* and *forced* should help to clarify the meanings.

Example 6-10

The relaxed series *RLC* circuit of Fig. 6-23a is excited at $t = 0$ by the sinusoidal source shown. Solve for the current $i(t)$ for $t > 0$.

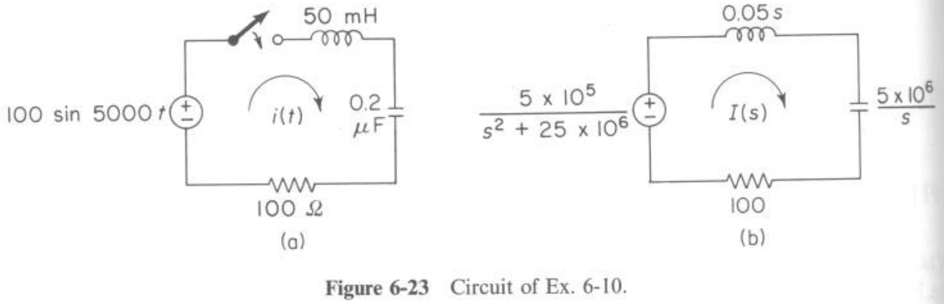

Figure 6-23 Circuit of Ex. 6-10.

Solution Although the mathematics will eventually reveal the type of response, a preliminary calculation should prove interesting. We will first calculate $R/2L$ and $1/\sqrt{LC}$.

$$\frac{R}{2L} = \frac{100}{2 \times 0.05} = 10^3 \tag{6-125a}$$

$$\frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.05 \times 0.2 \times 10^{-6}}} = 10^4 \tag{6-125b}$$

Since $R/2L < 1/\sqrt{LC}$, the circuit is underdamped and oscillatory. We have

$$\alpha = 10^3 \text{ nepers}^{\dagger} \tag{6-126}$$

$$\omega_0 = 10^4 \text{ rad/s} \tag{6-127}$$

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} = 9.95 \times 10^3 \text{ rad/s} \tag{6-128}$$

As a result of the relatively small amount of damping, the damped resonant frequency differs from the undamped resonant frequency by only 0.5%. As a matter of interest, the damped repetition frequency is $f_d = \omega_d/2\pi = 1548$ Hz. Notice that the natural damped frequency is about twice the frequency of the excitation. Again, we point out that these preliminary calculations are not absolutely necessary as the results will "fall out" of the math that follows.

The transformed circuit is shown in Fig. 6-23b. Using the impedance concept, we have

$$\begin{aligned} Z(s) &= 0.05s + 100 + \frac{5 \times 10^6}{s} \\ &= \frac{0.05s^2 + 100s + 5 \times 10^6}{s} \\ &= \frac{s^2 + 2000s + 10^8}{20s} \end{aligned} \tag{6-129}$$

The current is

$$I(s) = \frac{E(s)}{Z(s)} = \frac{10^7 s}{(s^2 + 25 \times 10^6)(s^2 + 2000s + 10^8)} \tag{6-130}$$

The poles due to the quadratic with three terms are

$$\begin{Bmatrix} s_1 \\ s_2 \end{Bmatrix} = -10^3 \pm j9.95 \times 10^3 \tag{6-131}$$

which agrees with our preliminary calculations.

We obtain the final desired result by finding the inverse transform of $I(s)$. Since one quadratic has imaginary roots and the other has complex roots, we

†Strictly speaking, both ω_0 and α have dimensions of time^{-1}, since radians are dimensionless. However, since ω_0 and α define different phenomena, it has been common practice to use a different unit for a damping constant. The common convention is to employ the unit *neper*.

may invert the function by applying the special formula of Section 5-7 individually to the two quadratic factors. The reader is invited to show that the result is

$$\begin{aligned} i(t) &= 0.133\epsilon^{-1000t} \sin (9.95 \times 10^3 t - 99.51^\circ) \\ &\quad + 0.132 \sin (5000t + 82.41^\circ) \end{aligned} \tag{6-132}$$

The response is seen to consist of a damped sinusoidal term whose frequency is the natural damped resonant frequency of the circuit, and an undamped sinusoid whose frequency is that of the excitation. The former term is transient in nature, whereas the latter term is the steady-state response. After the transient disappears, the steady-state or forced response is

$$i_{ss}(t) = 0.132 \sin (5000t + 82.41^\circ) \tag{6-133}$$

6-10 PARALLEL RLC CIRCUIT

In this section we turn our attention to another special case of a second-order circuit, the parallel *RLC* configuration. The reader familiar with the duality principle may recognize that the parallel *RLC* circuit is the dual of the series *RLC* circuit, and many of its properties could be deduced directly from the work of the preceding section. However, we develop the properties independently for the benefit of readers not familiar with this principle.

Consider the circuit shown in Fig. 6-24a, with no initial energy storage assumed, and its transform shown in (b). In analyzing parallel circuits such as this, it is better to assume current source excitation. Thus if the circuit is excited by a voltage source in series with a resistance, the source can be converted to an equivalent current source for analysis purposes. If the network is excited by a source that approximates an ideal voltage source, we could analyze the circuit by determining the individual currents by means of the basic voltage–current relationships for the elements.

The admittance of the network is given by

$$\begin{aligned} Y(s) &= sC + \frac{1}{R} + \frac{1}{sL} = \frac{s^2LC + sL/R + 1}{sL} \\ &= \frac{s^2 + s/RC + 1/LC}{s/C} \end{aligned} \tag{6-134}$$

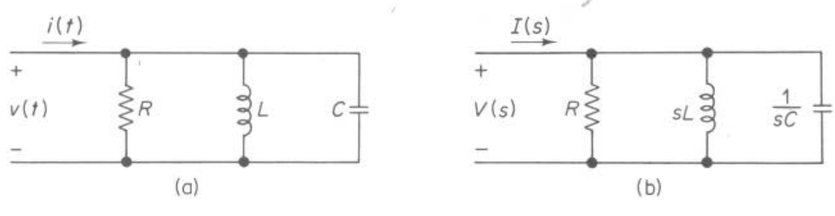

Figure 6-24 Parallel *RLC* circuit and its transform.

The impedance is

$$Z(s) = \frac{s/C}{s^2 + s/RC + 1/LC} \tag{6-135}$$

If a current $I(s)$ excites the network, the resulting voltage is

$$V(s) = Z(s)I(s) = \frac{sI(s)/C}{s^2 + s/RC + 1/LC} \tag{6-136}$$

As in the preceding section, let us direct our attention to the step response. Letting $I(s) = I/s$, we have

$$V(s) = \frac{I/C}{s^2 + s/RC + 1/LC} \tag{6-137}$$

We determine the poles by factoring the denominator polynomial.

$$\begin{Bmatrix} s_1 \\ s_2 \end{Bmatrix} = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}} \tag{6-138}$$

As in the case of the series RLC circuit, we have three possibilities:

1. *Overdamped*

$$\frac{1}{2RC} > \frac{1}{\sqrt{LC}} \tag{6-139}$$

2. *Critically damped*

$$\frac{1}{2RC} = \frac{1}{\sqrt{LC}} \tag{6-140}$$

3. *Underdamped*

$$\frac{1}{2RC} < \frac{1}{\sqrt{LC}} \tag{6-141}$$

In the underdamped case, we may define the undamped resonant frequency as

$$\omega_0 = \frac{1}{\sqrt{LC}} \tag{6-142}$$

Note that ω_0 is the same as in the series case. The damping factor in this case, however, is different and is

$$\alpha = \frac{1}{2RC} \tag{6-143}$$

As in the series case, the damped resonant frequency is

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} \tag{6-144}$$

The relative shapes and forms of the three possible types of voltage responses in this case are analogous to the current responses of the series *RLC* circuit of the preceding section.

Example 6-11

The relaxed circuit of Fig. 6-25a is excited at $t = 0$ by a pulse which approximates an impulse of area 100 V·s. Determine the voltage across the tuned circuit, $v(t)$, for $t > 0$.

Solution The first step in solving the problem is to rearrange the circuit in the simplest form for analysis. We do this by converting the impulse-voltage source to an impulse-current source and combining the two resistors. The result is shown in Fig. 6-25b and its transform is shown in (c). The admittance is

$$Y(s) = \frac{1}{20}s + \frac{1}{2} + \frac{4}{5s} = \frac{s^2 + 10s + 16}{20s} \tag{6-145}$$

$$Z(s) = \frac{20s}{s^2 + 10s + 16} \tag{6-146}$$

The voltage $V(s)$ is

$$V(s) = Z(s)I(s) = \frac{500s}{s^2 + 10s + 16} \tag{6-147}$$

The poles are

$$\begin{Bmatrix} s_1 \\ s_2 \end{Bmatrix} = -5 \pm \sqrt{25 - 16} = -5 \pm 3 = -2 \quad \text{and} \quad -8 \tag{6-148}$$

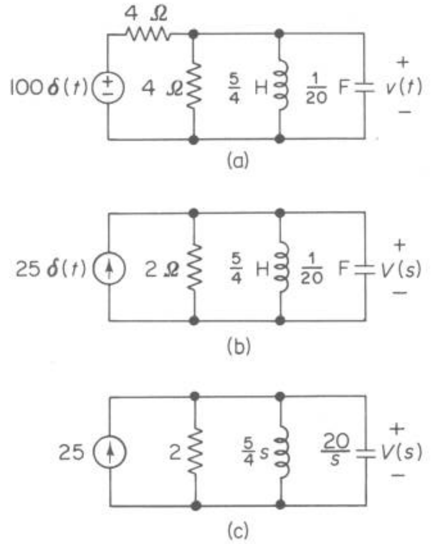

Figure 6-25 Circuits of Ex. 6-11.

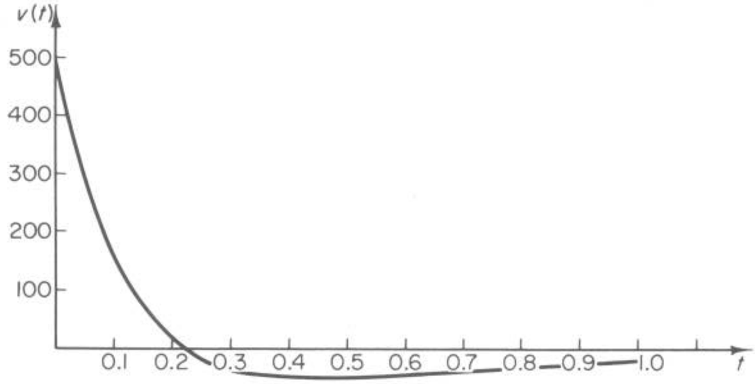

Figure 6-26 Responses of the circuit of Example 6-11.

Thus the response in this case is overdamped. We may write $V(s)$ as

$$V(s) = \frac{500s}{(s+2)(s+8)} = \frac{-500/3}{s+2} + \frac{2000/3}{s+8} \qquad (6\text{-}149)$$

The time response is thus

$$v(t) = \frac{500}{3}[4\epsilon^{-8t} - \epsilon^{-2t}] \qquad (6\text{-}150)$$

A sketch of the response is shown in Fig. 6-26.

6-11 REDUNDANCY AND HIGHER ORDER

In the preceding two sections we have considered two special common cases of second-order circuits, the series *RLC* and parallel *RLC* circuits. We saw that the response of such circuits could be either underdamped and oscillatory, overdamped, or critically damped. Practically speaking, component values are always subject to deviation, and thus the exact critically damped case probably occurs very rarely in practice. However, the relative shape of the time response curve varies so little as the system crosses the critically damped point that the assumption of such a response can be quite accurate.

There are many other forms for second-order *RLC* circuits, some of which are shown in Fig. 6-27. No generators are shown, as they could be connected in almost

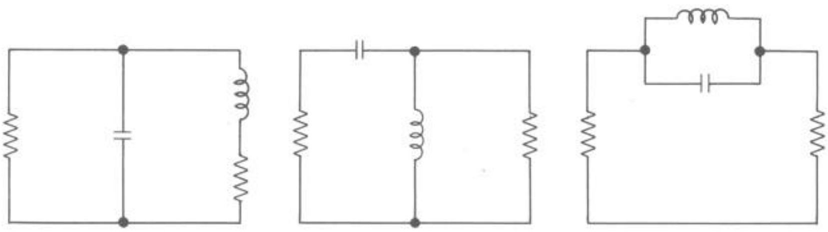

Figure 6-27 Some possible forms for second-order circuits.

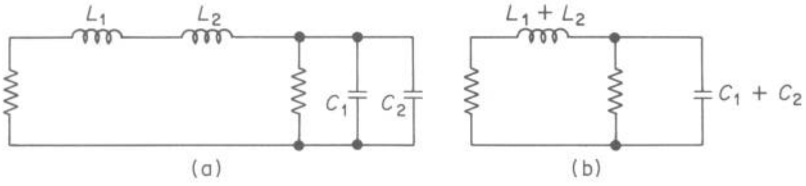

Figure 6-28 All illustration of redundancy in a circuit.

any imaginable manner. The reader might observe one important feature of all of these circuits which is identical with the series and parallel *RLC* circuits previously considered. All of these circuits contain one capacitor and one inductor. The question then arises as to whether or not there is a relationship between the number of energy-storage elements and the order of the system. The reader will recall that all first-order systems considered either contained only one reactive element or could be reduced to a system with only one reactive element. This reduction idea can be explained by the concept of redundancy. *A circuit containing two or more circuit components of the same type is said to be redundant if these components can be reduced to a single equivalent component under all conditions external to their terminals.*

Referring to Fig. 6-28a, we see that one of the two inductors, L_1 and L_2, is redundant since they can be reduced to a single inductor of value $L_1 + L_2$, as shown in (b). Similarly, the capacitors C_1 and C_2 can be reduced to a single capacitor of value $C_1 + C_2$, as shown in (b). Thus although this circuit might have appeared at first glance to be a higher-order circuit, it is really a second-order circuit after redundancy is considered.

However, the circuit of Fig. 6-29 is one in which no redundancy exists. Nowhere in the circuit are there two or more components that can be replaced by a single component.

The most common form of redundancy is either simple series or parellel connections of like components; thus such redundancy may be easily detected.

Some formal rules may now be stated: *The order of a circuit is the number of energy-storage elements after redundancy is removed.* Furthermore, *the order of the circuit is the order of the denominator polynomial of a given response when the only excitations in the circuit are impulse functions.* In effect, this latter statement says that poles due to excitations must not be counted in determining the order of a circuit from a transform function.

We may thus realize second- or higher-order circuits using only *RC* or *RL* forms. In other words, it is not necessary to have an *RLC* form to achieve a second- or higher-order circuit. Thus a circuit containing two nonredundant capacitors will be a second-order form. However, there is a basic limitation on the pole values of such a circuit containing only one type of energy-storage element. It can be shown that

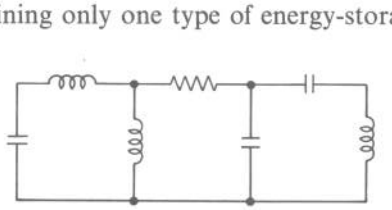

Figure 6-29 A circuit without redundancy.

the impulse response of a circuit without electronic feedback and containing only one type of energy-storage component (*i.e., either RC or RL circuits*) *will always be either overdamped or critically damped.* This is equivalent to saying that *the poles of the impulse response of such a circuit will always be real numbers and are never complex numbers.* In other words, to achieve an underdamped or oscillatory response in a circuit without feedback, it is necessary to employ both inductance and capacitance in the circuit. Since resistance is usually present, it is always realistic to refer to the latter form as an *RLC* circuit.

The reader might observe that we have excluded the possibility of electronic feedback. With feedback, it is possible to achieve an oscillatory response even with an *RC* form. Such considerations will be pursued somewhat in the next chapter.

In this chapter we have studied the general approaches for transforming a complete circuit so that it could be solved by transform methods. Furthermore, we have investigated in some detail first-order and second-order circuits and have obtained some complete solutions for such cases. In theory, we should be prepared to solve a more complex higher-order circuit by extending the basic network equations to such a problem.

In practice, the principal limitation to such higher-order solutions is the complexity of the computations. One of the principal problems that arises is the determination of the roots of the denominator polynomial for certain higher-order systems.

Modern technology provides the engineer and technologist with many computing facilities such as microcomputers and mainframes. The author feels that, beyond a certain order of complexity, it is best to depend, at least partially, on such aids. For example, most digital computers have standard numerical subroutines for determining the roots of higher-degree polynomials. Thus, during this phase of a problem solution, one could rely on a digital computer to actually determine the roots of a particular denominator polynomial.

6-12 SOLUTION FROM THE DIFFERENTIAL EQUATION

In many cases, the circuit analyst will have occasion to solve a given differential equation by transform methods. In this case, the circuit itself is not given, only the resulting differential equation and some initial conditions. In fact, the differential equation may not even represent a circuit but, perhaps, some other physical system entirely. In this section we consider how the transform process may be applied to such a representation.

The Laplace transform approach is best suited to ordinary linear differential equations of the constant-coefficient type. Such an equation of order m appears in the form

$$b_m \frac{d^m y}{dt^m} + b_{m-1} \frac{d^{m-1} y}{dt^{m-1}} + \cdots + b_0 y = f(t) \tag{6-151}$$

The equation is linear since there are no products of y and itself, or products of various derivatives, and because the coefficients are constant. The quantity $f(t)$ represents a functional combination of all excitations in the system. In Chapter 7 this quantity will be expanded to yield a different form. But for the purpose at hand, this form is adequate.

In working directly with circuit components in this chapter, we have never had to transform a derivative of any order higher than the first. The transform of the first derivative was stated in Chapter 5 to be

$$\mathscr{L}\left[\frac{dy}{dt}\right] = sY(s) - y(0) \tag{6-152}$$

To transform Eq. (6-151) we need an expression for the transform of higher-order derivatives. This can be readily deduced if we note that

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left[\frac{dy}{dt}\right] \tag{6-153}$$

In other words, the second derivative is the first derivative of the first derivative; hence operation (O-1) can be interpreted for the second derivative if $y(t)$ is replaced by dy/dt. Thus

$$\begin{aligned}\mathscr{L}\left[\frac{d^2y}{dt^2}\right] &= s\mathscr{L}\left[\frac{dy}{dt}\right] - y'(0) \\ &= s^2Y(s) - sy(0) - y'(0)\end{aligned} \tag{6-154}$$

where the prime signifies a derivative. A similar procedure for the third derivative yields

$$\begin{aligned}\mathscr{L}\left[\frac{d^3y}{dt^3}\right] &= s\mathscr{L}\left[\frac{d^2y}{dt^2}\right] - y''(0) \\ &= s^3Y(s) - s^2y(0) - sy'(0) - y''(0)\end{aligned} \tag{6-155}$$

In general,

$$\mathscr{L}\left[\frac{d^ky}{dt^k}\right] = s^kY(s) - s^{k-1}y(0) - s^{k-2}y'(0) - \cdots - y^{k-1}(0) \tag{6-156}$$

From the relationships above, it is clear that the transform of a kth-order derivative requires specification of the function and its first $k - 1$ derivatives at $t = 0$, or a total of k quantities. Since the highest-order derivative requires the most information, it is deduced that a differential equation of order m can be uniquely solved by a knowledge of the desired function and its first $m - 1$ derivatives evaluated at $t = 0$. The point $t = 0$ is usually interpreted to mean $t = 0^+$ in the sense that initial conditions are specified after the excitation is applied.

After the differential equation is transformed, $Y(s)$ may be determined by algebraic manipulation using the methods of Chapter 5. This phase of the problem is identical with the approach we have already considered.

In many cases, a system of simultaneous differential equations in terms of several variables will be given rather than a single differential equation. The same rule on the number of equations required holds as for the case of simultaneous algebraic equations. In other words, if there are N unknowns, it is necessary to have N simultaneous equations. Furthermore, each equation requires the specification of a number of initial conditions equal to its order.

Example 6-12

The response of a given physical system is described for $t > 0$ by the differential equation

$$4\frac{d^2y}{dt^2} + 24\frac{dy}{dt} + 32y = 100 \tag{6-157}$$

The initial values of y and dy/dt are

$$y(0) = +10 \tag{6-158}$$

and

$$y'(0) = -20 \tag{6-159}$$

Solve for $y(t)$, for $t > 0$, using Laplace transforms.

Solution Transforming the equation according to Eqs. (6-154) and (6-152), we have

$$4[s^2Y(s) - s(10) - (-20)] + 24[sY(s) - (10)] + 32Y(s) = \frac{100}{s} \tag{6-160}$$

After some rearrangement, we have

$$Y(s)[s^2 + 6s + 8] = \frac{25}{s} + 10s + 40 \tag{6-161}$$

or

$$Y(s) = \frac{25}{s(s^2 + 6s + 8)} + \frac{10s + 40}{s^2 + 6s + 8} \tag{6-162}$$

The reader is invited to complete the problem and obtain the solution:

$$y(t) = \frac{5}{8}(5 + 6\epsilon^{-2t} + 5\epsilon^{-4t}) \tag{6-163}$$

The reader might also check that the two initial conditions are satisfied by the solution.

Example 6-13

The initially relaxed circuit of Fig. 6-30a is excited at $t = 0$ by the dc voltage shown. Rather than transforming the circuit directly according to the early work of this chapter, write a set of simultaneous mesh differential equations in the time domain, deduce the necessary initial conditions, and solve for $i_2(t)$ by transform methods.

Solution The simultaneous mesh equations are

$$2i_1 + 8\int_0^t i_1\,dt + 2\left[\frac{di_1}{dt} - \frac{di_2}{dt}\right] = 10 \tag{6-164}$$

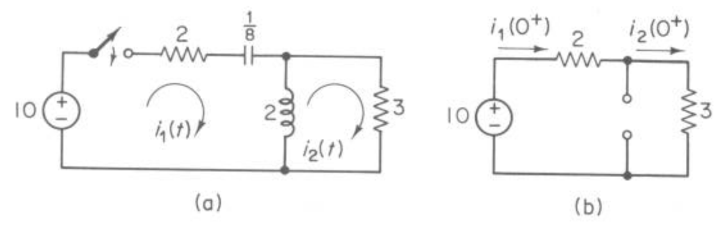

Figure 6-30 Circuit of Ex. 6-13.

and

$$2\left[\frac{di_2}{dt} - \frac{di_1}{dt}\right] + 3i_2 = 0 \tag{6-165}$$

Since the highest-order derivative in both equations is the first derivative, it is only necessary to know $i_1(0^+)$ and $i_2(0^+)$. These quantities are readily deduced from the initial circuit shown in Fig. 6-30b. We have

$$i_1(0^+) = i_2(0^+) = 2 \text{ A} \tag{6-166}$$

Using operation (O-2) and transform (T-2), we have

$$2I_1(s) + \frac{8}{s} I_1 + 2sI_1(s) - 4 - 2sI_2(s) + 4 = \frac{10}{s} \tag{6-167}$$

and

$$2sI_2(s) - 4 - 2sI_1(s) + 4 + 3I_2(s) = 0 \tag{6-168}$$

Simultaneous algebraic solution of these equations for $I_2(s)$ yields

$$I_2(s) = \frac{20s}{10s^2 + 22s + 24} = \frac{2s}{s^2 + 2.2s + 2.4} \tag{6-169}$$

Inversion of $I_2(s)$ is accomplished by means of the procedure of Section 5-7 since the poles are complex. The reader is invited to verify that

$$i_2(t) = 2.84\epsilon^{-1.1t} \sin(1.09t + 135.2°) \tag{6-170}$$

GENERAL PROBLEMS

6-1. An uncharged capacitance of 0.2 F is given. Write expressions for both the transform impedance and admittance. Draw the transform-domain equivalent circuit and label with the impedance value.

6-2. An uncharged capacitance of 0.5 μF is given. Write expressions for both the transform impedance and admittance. Draw the transform-domain equivalent circuit and label with the impedance value.

6-3. An unfluxed inductance of 4 H is given. Write expressions for both the transform impedance and admittance. Draw the transform-domain equivalent circuit and label with the impedance value.

6-4. An unfluxed inductance of 80 mH is given. Write expressions for both the transform impedance and admittance. Draw the transform-domain equivalent circuit and label with the impedance value.

6-5. For the charged capacitor of Fig. P6-5, draw both the Thévenin and Norton equivalent circuits in the transform domain.

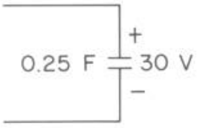

Figure P6-5

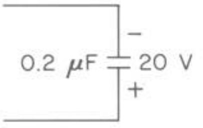

Figure P6-6

6-6. For the charged capacitor of Fig. P6-6, draw both the Thévenin and Norton equivalent circuits in the transform domain.

6-7. For the fluxed inductor of Fig. P6-7, draw both the Thévenin and Norton equivalent circuits in the transform domain.

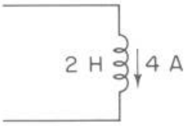

Figure P6-7

Figure P6-8

6-8. For the fluxed inductor of Fig. P6-8, draw both the Thévenin and Norton equivalent circuits in the transform domain.

6-9. For the circuit of Fig. P6-9, draw the transform-domain equivalent circuit in a form suitable for writing mesh current equations. Write the two transform-domain mesh current equations.

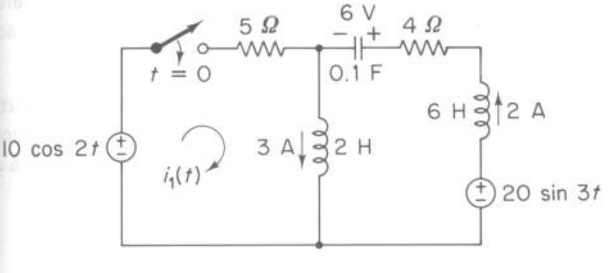

Figure P6-9

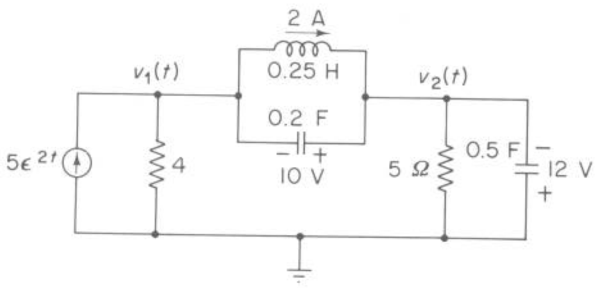

Figure P6-10

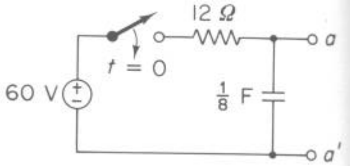

Figure P6-11

6-10. For the circuit of Fig. P6-10, draw the transform-domain equivalent circuit in a form suitable for writing node voltage equations. Write the two transform-domain node voltage equations.

6-11. **(a)** Obtain a transform Thévenin equivalent circuit at terminals *a-a'* for the circuit of Fig. P6-11 if the switch is closed at $t = 0$.

(b) Using the result of (a), determine the time-domain current $i(t)$ that would flow through a 6-Ω resistance connected to the terminals *a-a'* at $t = 0$.

6-12. Assume in Problem 6-11 that the switch has been closed a long time before $t = 0$ so that the circuit to the left of *a-a'* is in a steady-state condition. Repeat the analysis of Problem 6-11. It is assumed again that the 6-Ω resistance is connected at $t = 0$.

6-13. Determine the transform impedance $Z(s)$ seen at the terminals of the circuit of Fig. P6-13. Arrange as a ratio of polynomials.

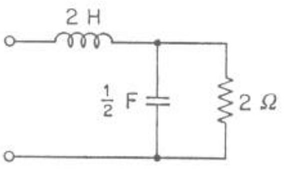

Figure P6-13

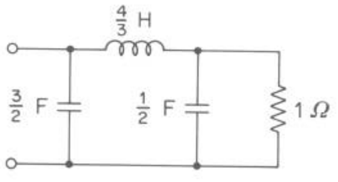

Figure P6-14

6-14. Determine the transform impedance $Z(s)$ seen at the terminals of the circuit of Fig. P6-14. Arrange as a ratio of polynomials

6-15. Assume that the circuit of Problem 6-13 is excited at $t = 0$ by a dc voltage source of value 10 V. Determine an expression for the transform current $I(s)$ that would flow from the source.

6-16. Assume that the circuit of Problem 16-14 is excited at $t = 0$ by a dc current source of value 5 A. Determine an expression for the transform voltage that would appear across the source.

In Problems 6-17 through 6-20, (a) write the forms of the inverse transforms of the functions with arbitrary coefficients assumed. In each case, $N(s)$ is some arbitrary numerator polynomial whose roots are assumed not to coincide with any roots of the denominator. (b) In each case, identify the transient and steady-state response terms.

6-17. $V(s) = \dfrac{N(s)}{s(s^2 + 4s + 3)}$

6-18. $I(s) = \dfrac{N(s)}{s(s^2 + 4s + 13)}$

6-19. $V(s) = \dfrac{N(s)}{(s^2 + 6s + 25)(s^2 + 4)}$

6-20. $I(s) = \dfrac{N(s)}{(s^2 + 2000s + 5 \times 10^6)(s^2 + 10^6)}$

6-21. For the circuit of Fig. P6-21, determine $i(t)$ and $v_C(t)$ if $e(t) = 10$ V. Identify transient and steady-state portions of the response.

6-22. Repeat the analysis of Problem 6-21 if $e(t) = 10 \sin 2t$.

6-23. Repeat the analysis of Problem 6-21 if $e(t) = 10 \cos 5t$.

6-24. Repeat the anaylsis of Problem 6-21 if $e(t) = 10\epsilon^{-5t}$.

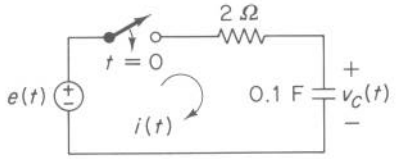

Figure P6-21

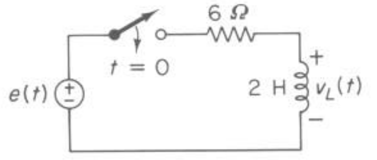

Figure P6-25

6-25. For the circuit of Fig. P6-25, determine $i(t)$ and $v_L(t)$ if $e(t) = 30$ V. Identify transient and steady-state portions of the response.

6-26. Repeat the analysis of Problem 6-25 if $e(t) = 30 \sin 2t$.

6-27. Repeat the analysis of Problem 6-25 if $e(t) = 30 \cos 4t$.

6-28. Repeat the analysis of Problem 6-25 if $e(t) = 30\delta(t)$.

6-29. The circuit of Fig. P6-29 is in a steady-state condition at $t = 0^-$. At $t = 0$, the switch is thrown to position 2. Solve for $v(t)$ for $t > 0$.

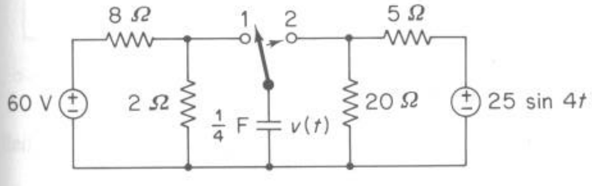

Figure P6-29

6-30. The circuit shown in Fig. P6-30a is excited at $t = 0$ by the narrow current pulse shown in (b).

(a) Solve for $v(t)$ by an exact analysis.

(b) Justify that the pulse may be approximated by an impulse source, and solve for the response due to this impulse. Compare the results.

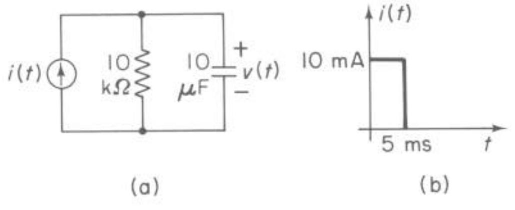

Figure P6-30

6-31. A series RLC circuit has $R = 2.4\ \Omega$, $L = 0.4$ H, and $C = 0.1$ F. Indicate whether the circuit is overdamped, critically damped, or underdamped. If the circuit is overdamped or critically damped, determine the damping factor(s). If underdamped, determine the damping factor, undamped natural frequency, and damped natural frequency.

6-32. Repeat the analysis of Problem 6-31 if $R = 5$ kΩ, $L = 2.5$ H, and $C = 0.4$ μF.

6-33. Repeat the analysis of Problem 6-31 if $R = 1.5$ kΩ, $L = 0.5$ H, and $C = 1$ μF.

6-34. The current in the series *RLC* circuit shown in Fig. P6-34 is know to be $i(t) = 2t$. Solve for $v(t)$ by any method. (*Hint:* This has been presented to refresh the reader's thinking on a basic point.)

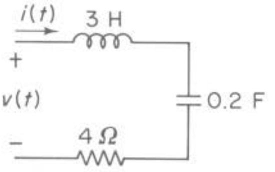

Figure P6-34

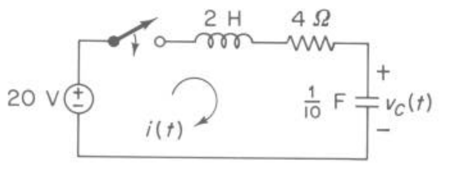

Figure P6-35

6-35. The switch in the circuit of Fig. P6-35 is closed at $t = 0$, and the circuit is initially relaxed. Determine the current $i(t)$ and the voltage $v_C(t)$.

6-36. The switch in the circuit of Fig. P6-36 is closed at $t = 0$, and the circuit is initially relaxed. Determine the current $i(t)$ and the voltage $v_C(t)$.

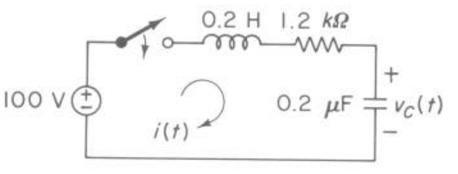

Figure P6-36

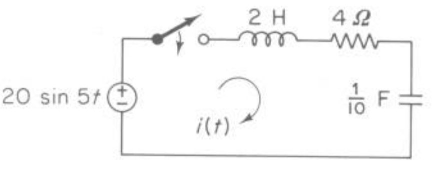

Figure P6-37

6-37. The switch in the circuit of Fig. P6-37 is closed at $t = 0$, and the circuit is initially relaxed. Determine the current $i(t)$.

6-38. The switch in the circuit of Fig. P6-38 is closed at $t = 0$, and the circuit is initially relaxed. Determine the current $i(t)$.

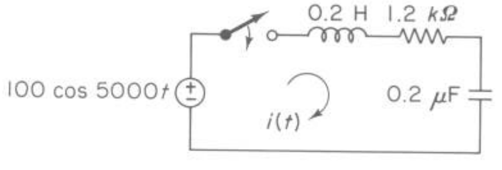

Figure P6-38

6-39. A parallel *RLC* circuit has $R = 1$ kΩ, $L = 2.5$ H, and $C = 0.1$ μF. Indicate whether the circuit is overdamped, critically damped, or underdamped. If the circuit is overdamped or critically damped, determine the damping factor(s). If underdamped, determine damping factor, undamped natural frequency, and damped natural frequency.

6-40. Repeat the analysis of Problem 6-39 if $R = 500$ Ω, $L = 0.1$ H, and $C = 2$ μF.

6-41. Repeat the analysis of Problem 6-39 if $R = 20$ kΩ, $L = 0.1$ H, and $C = 0.001$ μF.

6-42. The voltage across the parallel *RLC* circuit shown in Fig. P6-42 is known to be $v(t) = 40t$ Solve for $i(t)$ by any method. (*Hint:* This has been presented to refresh the reader's thinking on a basic point.)

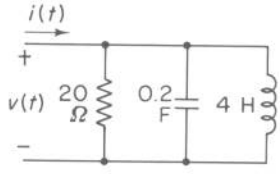

Figure P6-42

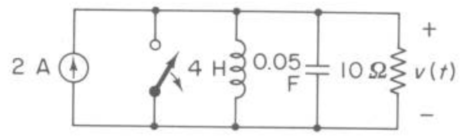

Figure P6-43

6-43. The switch in the circuit of Fig. P6-43 is opened at $t = 0$, and the circuit is initially relaxed. Determine the voltage $v(t)$.

6-44. The switch in the circuit of Fig. P6-44 is opened at $t = 0$, and the circuit is initially relaxed. Determine the voltage $v(t)$.

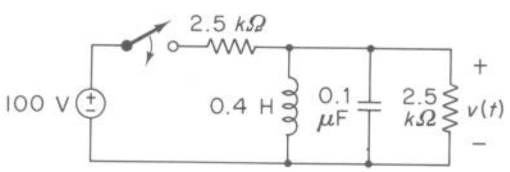

Figure P6-44

6-45. Determine the order of the circuit shown in Fig. P6-45.

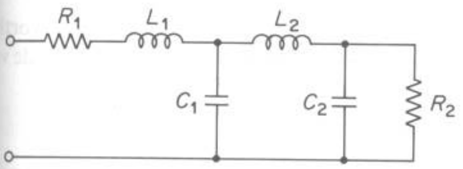

Figure P6-45

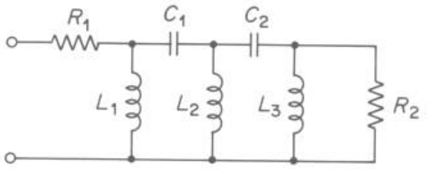

Figure P6-46

6-46. Determine the order of the circuit shown in Fig. P6-46.

6-47. Determine the order of the circuit shown in Fig. P6-47.

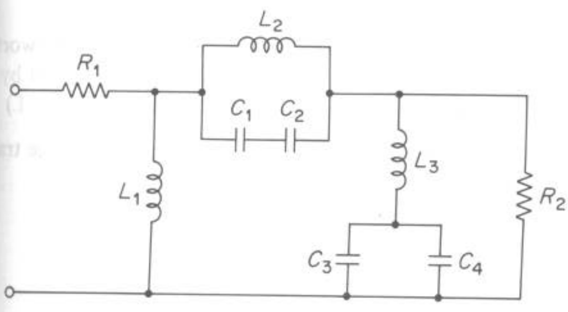

Figure P6-47

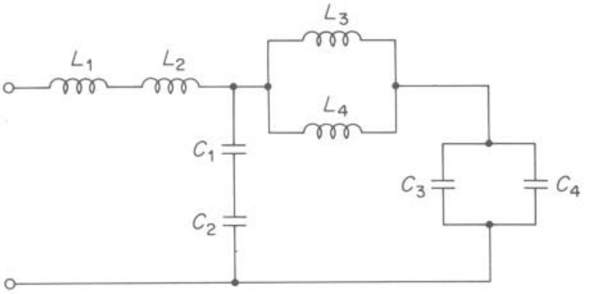

Figure P6-48

6-48. Determine the order of the circuit shown in Fig. P6-48.

6-49. The circuit of Fig. P6-49 is one form of a low-pass second-order passive Butterworth filter normalized to a cutoff frequency of 1 rad/s. The circuit is excited at $t = 0$ by the dc voltage shown. Determine the output voltage $v(t)$.

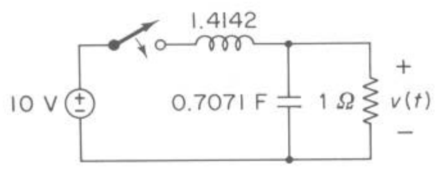

Figure P6-49

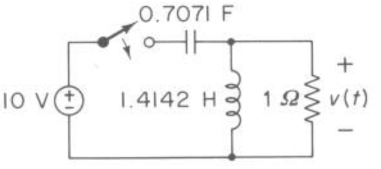

Figure P6-50

6-50. The circuit of Fig. P6-50 is one form of a high-pass second-order passive Butterworth filter normalized to a cutoff frequency of 1 rad/s. The circuit is excited at $t = 0$ by the dc voltage shown. Determine the output voltage $v(t)$.

6-51. The circuit of Fig. P6-51 is one form of a low-pass third-order passive Butterworth filter normalized to a cutoff frequency of 1 rad/s. The circuit is excited at $t = 0$ by the dc voltage shown. Determine the output voltage $v(t)$. (*Hint:* One pole is $s = -1$.)

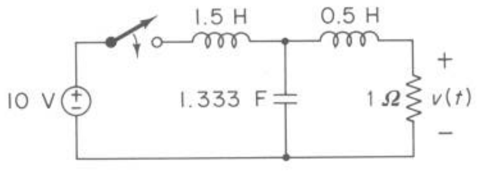

Figure P6-51

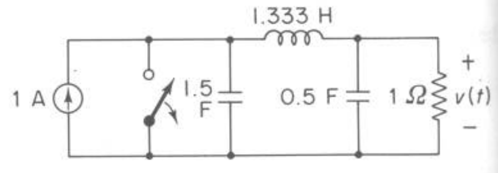

Figure P6-52

6-52. The circuit of Fig. P6-52 is one form of a low-pass third-order passive Butterworth filter normalized to a cutoff frequency of 1 rad/s. The circuit is excited at $t = 0$ by the dc current shown. Determine the output voltage $v(t)$. (*Hint:* One pole is $s = -1$.)

In Problems 6-53 through 6-56, solve the differential equations given using Laplace transform methods. Note the initial conditions stated.

6-53. $2\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 4y = 24$

$y(0) = 12$, $y'(0) = -4$

6-54. $4\frac{dy}{dt} + 12y = 20 \sin 4t$

$y(0) = -4$

6-55. $2\frac{dy}{dt} + 8y + 26\int_0^t y\,dt = 10\cos t$

$y(0) = 0$

6-56. $\frac{d^2y}{dt^2} + 3000\frac{dy}{dt} + 2 \times 10^6 y = 10^8$

$y(0) = 75,\ y'(10) = 10^4$

6-57. For the circuit of Problem 6-35 (Fig. P6-35), solve for the current $i(t)$ by first writing a time-domain mesh current integrodifferential equation, specifying appropriate initial conditions, and solving the equation using Laplace transforms.

6-58. For the circuit of Problem 6-44 (Fig. P6-44), solve for the voltage $v(t)$ by first writing a time-domain node voltage integrodifferential equation, specifying appropriate initial conditions, and solving the equation using Laplace transforms.

DERIVATION PROBLEMS

6-59. The derivation of transform capacitive impedance was achieved in the text with operation (O-2) of Table 5-2. Provide an alternate derivation by first starting with the derivative relationship for current in terms of voltage and applying operation (O-1) of Table 5-2.

6-60. The derivation of transform inductive impedance was achieved in the text with operation (O-1) of Table 5-2. Provide an alternate derivation by first starting with the integral relationship for current in terms of voltage and applying operation (O-2) of Table 5-2.

6-61. Under certain conditions, it is possible for a given initial condition to result in a total cancellation of the natural or transient response. Consider the *RC* circuit of Fig. P6-61 with the capacitor initially charged to a voltage V_0 as shown. Assume a sinusoidal excitation at $t = 0$. Show that the transient response completely vanishes if the following condition is met:

$$V_0 = \frac{\omega RCV_p}{1 + \omega^2 R^2 C^2}$$

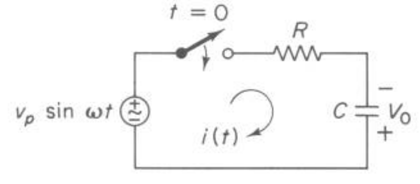

Figure P6-61

6-62. For the parallel *RLC* circuit of Section 6-10 excited by a step current of magnitude *I*, show that the voltage for the underdamped case is

$$v(t) = \frac{I}{\omega_d C}\epsilon^{-\alpha t}\sin\omega_d t$$

6-63. Consider the current response of the critically damped series *RLC* circuit as given by Eq. (6-118).

(a) Show that the maximum value of the current occurs at a time t_m given by

$$t_m = \frac{2L}{R}$$

(b) Show that the maximum value of the current I_m is given by

$$I_m = 0.7358 \frac{E}{R}$$

6-64. Consider the current response of the underdamped series RLC circuit as given by Eq. (6-124). Show that the first maximum of the current occurs at a time t_m given by

$$t_m = \frac{1}{\omega_d} \tan^{-1} \frac{\omega_d}{\alpha}$$

COMPUTER PROBLEMS

6-65. Any response $y(t)$ of a first-order circuit with a sinusoidal input may be expressed as

$$y(t) = A\epsilon^{-t/\tau} + B \sin(\omega t + \theta)$$

Write a computer program to evaluate $y(t)$ at intervals of Δt in the range from t_1 to t_2.

Input data: $A, B, \tau, \omega, \theta, t_1, t_2, \Delta t$

Output data: t, $y(t)$

(*Note:* The sinusoidal function should be programmed to accept angles in radians, and θ should be expressed in radians.)

6-66. Write a computer program to evaluate the current $i(t)$ for the step response of a series RLC circuit at intervals of Δt in the range from t_1 to t_2. Means should be provided to test whether the response is overdamped, critically damped, or underdamped and to branch to appropriate segments as required.

Input data: $E, R, L, C, t_1, t_2, \Delta t$

Output data: t, $i(t)$, a statement to indicate form of response.

(*Note:* Underdamped response should be programmed to manipulate angle in radians.)