A Crash Course on Solving Equations with Logarithms and Exponentials

Rules:

$$\log_b M + \log_b N = \log_b MN \qquad b^{x+y} = b^x \cdot b^y$$

$$\log_b M - \log_b N = \log_b \left(\frac{M}{N}\right) \qquad b^{x-y} = \frac{b^x}{b^y}$$

$$k \cdot \log_b M = \log_b M^k \qquad b^{x \cdot k} = (b^x)^k$$

Relationships:

$$b^E = A \iff \log_b A = E \qquad \log_b M = \frac{\log_a M}{\log_a b}$$

Example: Expand $\ln \sqrt[3]{\frac{a^2 b}{b^2 c}}$.

Solution:

$$\ln \sqrt[3]{\frac{a^2 b}{b^2 c}} = \ln \left(\frac{a^2 b}{b^2 c}\right)^{\frac{1}{3}} = \ln \left(\frac{a^2}{bc}\right)^{\frac{1}{3}} = \frac{1}{3} \ln \left(\frac{a^2}{bc}\right) = \frac{1}{3}[\ln a^2 - \ln bc]$$

$$= \frac{1}{3}[2 \ln a - (\ln b + \ln c)] = \frac{1}{3}[2 \ln a - \ln b - \ln c]$$

$$= \frac{2}{3} \ln a - \frac{1}{3} \ln b - \frac{1}{3} \ln c$$

Example: Compress the logarithm $4 \log_5 3 + \log_5 (x-1) - 3 \log_5 x$ into a single logarithm.

Solution:

$$4 \log_5 3 + \log_5 (x-1) - 3 \log_5 x = \log_5 3^4 + \log_5 (x-1) - \log_5 x^3$$

$$= \log_5 3^4 (x-1) - \log_5 x^3 = \log_5 \frac{3^4 (x-1)}{x^3} = \log_5 \frac{81(x-1)}{x^3}$$

Example: Solve for x in $\log_3 x + \log_3(x-7) = \log_3 8$

Solution:

$$\log_3 x + \log_3(x-7) - \log_3 8 = 0$$

$$\log_3 \frac{x(x-7)}{8} = 0$$

$$3^0 = \frac{x(x-7)}{8}$$

$$8 = x(x-7)$$

$$8 = x^2 - 7x$$

$$0 = x^2 - 7x - 8$$

$$0 = (x-8)(x+1)$$

$$x = 8$$

$$note: \; x \neq -1 \; because \; \log_3(-1) \; doesn't \; exist$$

Example: Solve for x in $3^{x+7} = 5^{2x-3}$

Solution:

$$\ln 3^{x+7} = \ln 5^{2x-3}$$

$$(x+7)\ln 3 = (2x-3)\ln 5$$

$$x\ln 3 + 7\ln 3 = 2x\ln 5 - 3\ln 5$$

$$7\ln 3 + 3\ln 5 = 2x\ln 5 - x\ln 3$$

$$7\ln 3 + 3\ln 5 = x(2\ln 5 - \ln 3)$$

$$\frac{7\ln 3 + 3\ln 5}{2\ln 5 - \ln 3} = x$$

Which is the same as $x = 5.904265929$