The Formal Definition of the Limit

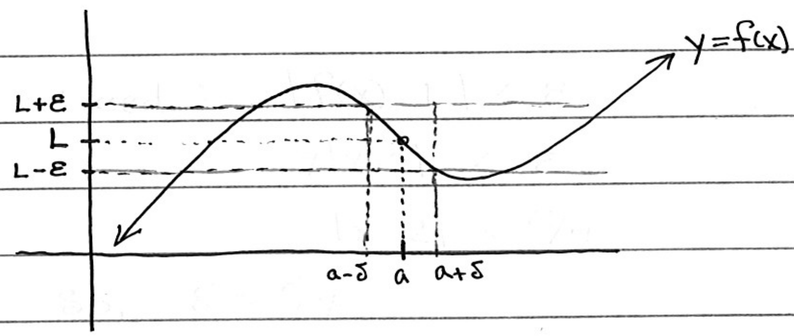
Definition:

Let f be a function defined on some open interval containing a, except possibly at $x=a$. Then $\lim_{x \to a} f(x) = L$

$$\text{if } \forall \varepsilon > 0 \;\exists\; \delta > 0 \ni |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

<u>Note</u>: both a & L not $\pm\infty$

Illustration:



means:

you want to find L to a certain accuracy

- pick $\varepsilon > 0$
- find $\delta > 0$
- choose x_a where $a - \delta < x_a < a + \delta$
- then $f(x_a)$ will be $L - \varepsilon < f(x_a) < L + \varepsilon$

So, $f(x_a)$ is w/i a certain accuracy of L

example: Prove $\lim_{x \to 2} (7x-1) = 13$

want to find $\delta > 0$ where for any $\varepsilon > 0$
if $|x-2| < \delta$ implies that $|(7x-1)-13| < \varepsilon$.

format: δ will be in terms of ε

Method:

$$|f(x)-L| = |(7x-1)-13| = |7x-14|$$
$$= |7(x-2)| = 7|x-2|$$

need: $|f(x)-L| < \varepsilon$

$$7|x-2| < \varepsilon$$
$$|x-2| < \varepsilon/7$$

so, $\delta = \varepsilon/7$

$\therefore \forall \varepsilon > 0$ if $|x-2| < \varepsilon/7$ then $|(7x-1)-13| < \varepsilon$

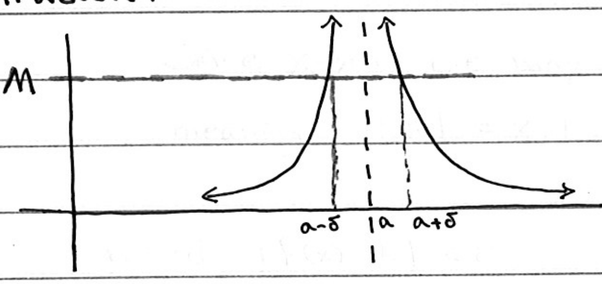
At Verticle Asymptotes:

Definition:

Let f be a function defined on some open interval containing a, except possibly at $x=a$.
Then $\lim_{x\to a} f(x) = \infty$ means that

for every $M>0$ $\exists\ \delta>0$ $|x-a|<\delta \Rightarrow f(x)>M$

Illustration:



- pick M (as large as you like)
- find δ
- choose x_a where $a-\delta < x_a < a+\delta$
- Then $f(x_a) > M$

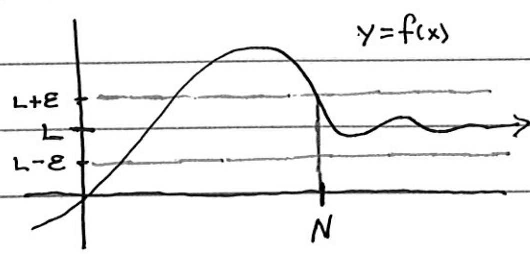
At Ends:

Definition:

Let f be a function defined on (a,∞)
Then $\lim_{x\to\infty} f(x) = L$ means that

$\forall \varepsilon>0\ \exists N>a \ni x>N \Rightarrow |f(x)-L|<\varepsilon$

Illustration:



- pick $\varepsilon>0$
- find N
- choose $x_N \ni x_N>N$
- Then $f(x_N)$ will be $L-\varepsilon < f(x_N) < L+\varepsilon$

example: Prove that $\lim_{x\to\infty} \frac{x}{x+1} = 1$

want to find N where for any $\varepsilon > 0$

$x > N$ implies that $\left| \frac{x}{x+1} - 1 \right| < \varepsilon$

format: N will be in terms of ε

Method: $|f(x) - L| = \left| \frac{x}{x+1} - 1 \right| = \left| \frac{x}{x+1} - \frac{x+1}{x+1} \right|$

$= \left| \frac{-1}{x+1} \right| = \frac{1}{|x+1|} = \frac{1}{x+1} < \frac{1}{x}$

Since $x \to \infty$, we may assume $x > 0$

meaning $|x+1| = x+1$ and $\frac{1}{x+1} < \frac{1}{x}$

need $|f(x) - L| < \varepsilon$

$\frac{1}{x} < \varepsilon$

$x > \frac{1}{\varepsilon}$

pick $N = 1/\varepsilon$

$\therefore$ $\forall \varepsilon > 0$ if $x > 1/\varepsilon$ then $\left| \frac{x}{x+1} - 1 \right| < \varepsilon$