Key

Homework Set 16

The Formal Definition of a Limit

(sections 1.3 & 1.6)

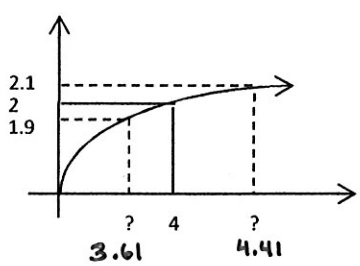
1. State the formal $\epsilon - \delta$ definition of a limit. (Hint: there are three versions.)

- $\lim_{x \to a} f(x) = L$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that
  if $|x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon$

- $\lim_{x \to a} f(x) = \infty$ if for every positive number M there is a $\delta > 0$ such that
  if $|x-a| < \delta \Rightarrow f(x) > M$

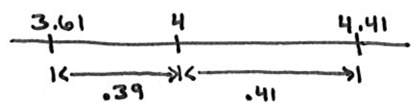
- $\lim_{x \to \infty} f(x) = L$ if for every $\varepsilon > 0$ there is a number (positive) N such that
  if $x > N \Rightarrow |f(x) - L| < \varepsilon$

2. To illustrate $\lim_{x\to 4} \sqrt{x} \overset{=2}{,}$ use the graph (below) of $f(x) = \sqrt{x}$ to find a number δ such that $|x-4| < \delta$ guarantees that $|\sqrt{x} - 2| < 0.1$



$2 + 0.1 = 2.1 = \sqrt{x} \Rightarrow x = 4.41$

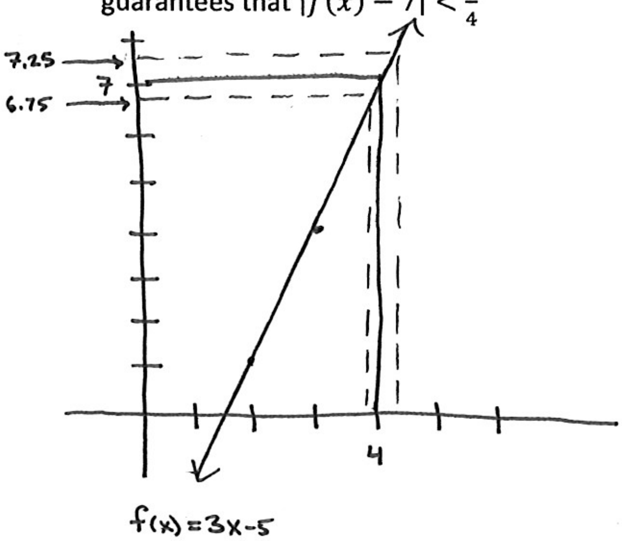
$2 - 0.1 = 1.9 = \sqrt{x} \Rightarrow x = 3.61$



need $\delta < .39$

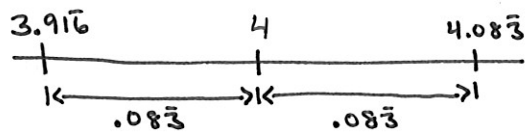
pick $\delta = .35$

3. Illustrate the formal $\epsilon - \delta$ definition of a limit applied to $\lim_{x\to 4}(3x-5) = 7$ with a diagram (see figure 15 with example 9 in the textbook). Find a number δ such that $|x-4| < \delta$ guarantees that $|f(x) - 7| < \frac{1}{4}$



$7 + 1/4 = 7.25 = 3x - 5 \Rightarrow 4.08\bar{3} = 49/12 = x$

$7 - 1/4 = 6.75 = 3x - 5 \Rightarrow 3.91\bar{6} = 47/12 = x$



need $\delta < .08\bar{3} = \frac{1}{12}$

pick $\delta = .08$

4. Use the formal $\epsilon-\delta$ definition of a limit to show that $\lim_{x\to 4}(3x-5)=7$.

assume $|f(x)-7|<\varepsilon$

by assumption

$|f(x)-7| = |(3x-5)-7| = |3x-12| = |3(x-4)| = 3|x-4| < \varepsilon$

then we need $|x-4|<\varepsilon/3 \Rightarrow$ choose $\delta=\varepsilon/3$

$\therefore$ if $|x-4|<\varepsilon/3 \Rightarrow |(3x-5)-7|<\varepsilon$ for any $\varepsilon>0$

which means that $\lim_{x\to 4}(3x-5)=7$

5. To illustrate $\lim_{x\to\infty}\frac{2x}{x-1}=2$, find a number N such that $x>N$ guarantees that $\left|\frac{2x}{x-1}-2\right|<\frac{1}{100}$ (hint: you may want to look at the graph of $f(x)=\frac{2x}{x-1}$). What is the smallest x value such that $f(x)$ is within two decimal places of the limit?

$\left|\frac{2x}{x-1}-2\right|<\frac{1}{100}$

$\left|\frac{2x}{x-1}-\frac{2(x-1)}{x-1}\right|<\frac{1}{100}$

$\left|\frac{2}{x-1}\right|<100$

$200<|x-1|$

since $x\to\infty$, we look at

$200<x-1$

$199<x$

So, $N=199$

The smallest x-value is $x=200$ such that $\frac{2x}{x-1}$ is w/i 2 decimal places of 2.

6. Use the formal $\epsilon-\delta$ definition of a limit to show that $\lim_{x\to\infty}\frac{2x}{x-1}=2$.

assume $|f(x)-2|<\varepsilon$

by assumption

$\left|\frac{2x}{x-1}-2\right| = \left|\frac{2}{x-1}\right| = \frac{2}{|x-1|} < \varepsilon$

then we need $\frac{2}{\varepsilon}<|x-1|$

now for $x\to\infty$, $x>|x-1|$ so $x>2/\varepsilon$

choose $N=2/\varepsilon$

$\therefore$ if $x>2/\varepsilon \Rightarrow \left|\frac{2x}{x-1}-2\right|<\varepsilon$ for any $\varepsilon>0$

which means that $\lim_{x\to\infty}\frac{2x}{x-1}=2$