Key

Homework Set 17

The (Limit) Definition of the Derivative

(sections 2.1 & 2.2)

1. State the formal definition of the derivative of a function. (Hint: this includes a formula.)

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

Compute the derivative for each of the following functions using the definition of the derivative.

2. $f(x) = 3x^2 - x + 7$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{[3(x+h)^2-(x+h)+7]-[3x^2-x+7]}{h}$$

$$= \lim_{h\to 0} \frac{(3x^2+6xh+h^2-x-h+7)-(3x^2-x+7)}{h} = \lim_{h\to 0} \frac{6xh+h^2-h}{h}$$

$$= \lim_{h\to 0} (6x+h-1) = 6x-1$$

3. $g(t) = \frac{2t-1}{t+5}$

$$g'(t) = \lim_{h\to 0} \frac{g(t+h)-g(t)}{h} = \lim_{h\to 0} \frac{\frac{2(t+h)-1}{t+h+5} - \frac{2t-1}{t+5}}{h} = \lim_{h\to 0} \frac{1}{h}\left[\frac{2t+2h-1}{t+h+5} - \frac{2t-1}{t+5}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{(2t+2h-1)(t+5)}{(t+h+5)(t+5)} - \frac{(2t-1)(t+h+5)}{(t+5)(t+h+5)}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{(\cancel{2t^2}+\cancel{2ht}-\cancel{t}+\cancel{10t}+10h-\cancel{5})-(\cancel{2t^2}+\cancel{2th}+\cancel{10t}-\cancel{t}-h-\cancel{5})}{(t+h+5)(t+5)}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\cdot\frac{11h}{(t+h+5)(t+5)} = \lim_{h\to 0} \frac{11}{(t+h+5)(t+5)}$$

$$= \frac{11}{(t+5)^2}$$

4. $f(x) = 4$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{4-4}{h}$$

$$= \lim_{h\to 0} \frac{0}{h} = \lim_{h\to 0} 0 = 0$$

5. $h(x) = \sqrt{2x+1}$

$$h'(x) = \lim_{h\to 0} \frac{\sqrt{2(x+h)+1} - \sqrt{2x+1}}{h} \cdot \frac{\sqrt{2x+2h+1} + \sqrt{2x+1}}{\sqrt{2x+2h+1} + \sqrt{2x+1}}$$

$$= \lim_{h\to 0} \frac{(2x+2h+1)-(2x+1)}{h\left[\sqrt{2x+2h+1} + \sqrt{2x+1}\right]}$$

$$= \lim_{h\to 0} \frac{2}{\sqrt{2x+2h+1} + \sqrt{2x+1}} = \frac{2}{2\sqrt{2x+1}} = \frac{1}{\sqrt{2x+1}}$$

6. $g(t) = \frac{1}{\sqrt{t}}$

$$g'(t) = \lim_{h\to 0} \frac{\frac{1}{\sqrt{t+h}} - \frac{1}{\sqrt{t}}}{h} = \lim_{h\to 0} \frac{\sqrt{t} - \sqrt{t+h}}{h\left[\sqrt{t}\sqrt{t+h}\right]} \cdot \frac{\sqrt{t} + \sqrt{t+h}}{\sqrt{t} + \sqrt{t+h}}$$

$$= \lim_{h\to 0} \frac{t-(t+h)}{h\left[t\sqrt{t+h} + (t+h)\sqrt{t}\right]} = \lim_{h\to 0} \frac{-1}{t\sqrt{t+h} + (t+h)\sqrt{t}}$$

$$= \frac{-1}{2t\sqrt{t}}$$

Each of the following limits represents the derivative of some function f at some number a. Find the function f and the number a.

7. $$\lim_{h\to 0} \frac{\sqrt[3]{27+h} - 3}{h}$$

$a = 27$

$f(x) = \sqrt[3]{x}$

8. $$\lim_{x\to 1} \frac{x^4 + x - 2}{x - 1}$$

$a = 1$

$f(x) = x^4 + x$

9. $$\lim_{h\to 0} \frac{(2+h)^3 - 8}{h}$$

$a = 2$

$f(x) = x^3$

10. $$\lim_{x\to \pi/4} \frac{\tan x - 1}{x - \pi/4}$$

$a = \pi/4$

$f(x) = \tan x$