Homework Set 27

Related Rates and Error-checking with Differentials

(sections 2.7 & 2.8)

1. Let $4x^2 + 9y^2 = 36$ where both x and y are functions of time t.
   a. If $\frac{dy}{dx} = \frac{1}{3}$, find $\frac{dx}{dt}$ when $x = 2$ and $y = \frac{2}{3}\sqrt{5}$.

$$8x \cdot \frac{dx}{dt} + 18y \cdot \frac{dy}{dt} = 0$$

$$8 \cdot 2 \cdot \frac{dx}{dt} + 18 \cdot \frac{2}{3}\sqrt{5} \cdot \frac{1}{3} = 0$$

$$16 \cdot \frac{dx}{dt} = -4\sqrt{5}$$

So, $\frac{dx}{dt} = -\frac{\sqrt{5}}{4}$

   b. If $\frac{dx}{dt} = 3$, find $\frac{dy}{dt}$ when $x = -2$ and $y = \frac{2}{3}\sqrt{5}$.

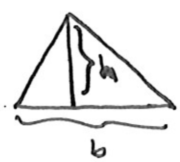
$$8x \cdot \frac{dx}{dt} + 18y \cdot \frac{dy}{dt} = 0$$

$$8(-2) \cdot 3 + 18 \cdot \frac{2}{3}\sqrt{5} \cdot \frac{dy}{dt} = 0$$

$$12\sqrt{5} \cdot \frac{dy}{dt} = 48$$

So, $\frac{dy}{dt} = \frac{4}{\sqrt{5}}$

2. The altitude of a triangle is increasing at a rate of 1 cm/min while the area of the triangle is increasing at a rate of 2 cm^2/min. At what rate is the base of the triangle changing when the altitude is 10 cm and the area is 100 cm^2?



Given : $\frac{dh}{dt} = 1$ cm/min , $\frac{dA}{dt} = 2$ cm²/min

Find : $\frac{db}{dt} = ?$ when $h = 10$cm and $A = 100$cm²

$$A = \frac{1}{2}hb$$

$$\frac{dA}{dt} = \frac{1}{2} \cdot \frac{dh}{dt} b + \frac{1}{2}h \cdot \frac{db}{dt}$$

$$2\,\frac{cm^2}{min} = \frac{1}{2}\left(1\,\frac{cm}{min}\right)(20\,cm) + \frac{1}{2}(10\,cm)\left(\frac{db}{dt}\right)$$

$$2 = 10 + 5 \cdot \frac{db}{dt} \implies \text{So, } \frac{db}{dt} = -\frac{8}{5}\ \text{cm/min}$$

note: altitude = height

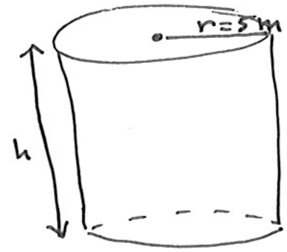
$A = \frac{1}{2}hb$

$100 = \frac{1}{2}(10)b$

$20 = b$

3. A cylindrical tank with a radius of 5 meters is filled with water at a rate of 3 m^3/sec. How fast is the height of the water rising?

$V = \pi r^2 h$



$$V = \pi r^2 h$$

$$V = 25\pi h$$

$$\frac{dV}{dt} = 25\pi \cdot \frac{dh}{dt}$$

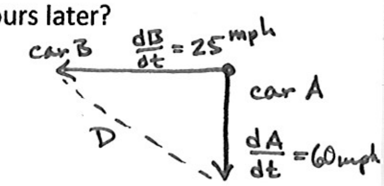
$$3 = 25\pi \cdot \frac{dh}{dt}$$

So, $\frac{dh}{dt} = \frac{3}{25\pi}$ m/sec

$\frac{dV}{dt} = 3$ m³/sec

find $\frac{dh}{dt}$

4. Two cars start moving from the same point. One travels south at 60 mph and the other travels west at 25 mph. At what rate is the distance between the cars increasing two hours later?



$$D^2 = A^2 + B^2$$

$$2D \cdot \frac{dD}{dt} = 2A \cdot \frac{dA}{dt} + 2B \cdot \frac{dB}{dt}$$

$$2(130) \cdot \frac{dD}{dt} = 2(120) \cdot 60 + 2(50) \cdot 25$$

$$260 \frac{dD}{dt} = 16900$$

$$\frac{dD}{dt} = 65 \text{ mph}$$

$D^2 = B^2 + A^2$

after 2 hrs:

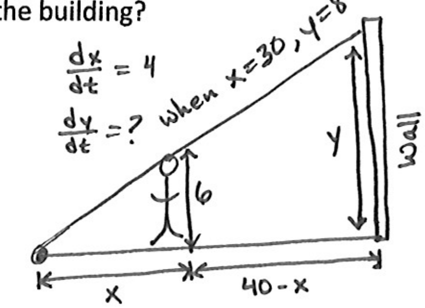
$A = 120$ mi

$B = 50$ mi

$D = 130$ mi

Find $\frac{dD}{dt}$

5. A spotlight on the ground shines on a wall 40 feet away. If a man 6 feet tall walks from the spotlight towards the building at a speed of 4 feet per second, how fast is the length of his shadow on the building decreasing when he is 10 feet away from the building?



Similar triangles:

$$\frac{6}{x} = \frac{y}{40}$$

$$xy = 240$$

$$\frac{dx}{dt} \cdot y + x \cdot \frac{dy}{dt} = 0$$

$$4 \cdot 8 + 30 \cdot \frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = -32/30 = -16/15$$

when $40 - x = 10 \Rightarrow x = 30$

when $x = 30$

$\frac{6}{30} = \frac{y}{40} \Rightarrow y = 8$

6. The edge of a cube was found to be 30 cm with a possible error of in measurement of 0.1 cm. Use differentials to estimate the maximum possible error, the relative error, and the percentage error when computing:

a. the volume of the cube

$$V = s^3$$

$$dV = 3s^2 \cdot ds$$

max error:

$$dV = 3(30)^2(.1)$$

$$= 270 \text{ cm}^3$$

relative error:

$$\frac{dV}{V} = \frac{3s^2 ds}{s^3} = \frac{3ds}{s} = \frac{3(.1)}{30} = .01$$

percentage error:

$.01 \longrightarrow 1\%$

b. the surface area of the cube

$$A = 6s^2$$

$$dA = 12s \cdot ds$$

max error:

$$dA = 12(30)(.1)$$

$$= 36 \text{ cm}^2$$

relative error:

$$\frac{dA}{A} = \frac{12s\,ds}{6s^2} = \frac{2ds}{s} = \frac{2(.1)}{30} = .00\bar{6}$$

percentage error:

$.00\bar{6} \longrightarrow .\bar{6}\%$

7. Use differentials to estimate the amount of paint needed to apply a coat of paint 0.05 cm thick to a hemispherical dome with a diameter of 50 meters.

S.A. of sphere : $4\pi r^2$

$A = 2\pi r^2$

$dA = 4\pi r \cdot dr$

$dA = 4\pi(25)(.0005)$

$= .05\pi \text{ m}^2$

$= .1570796 \text{ m}^2$

OR $= 1570.796 \text{ cm}^2$

$r = 25$ m

$SA = 4\pi r^2$ (sphere)

$dr = 0.05$ cm

$= .0005$ m

8. If a current I passes through a resistor with resistance R, Ohm's Law states that the resulting voltage drop is $V = RI$. If V is constant and R is measured with a certain error, use differentials to show that the relative error in calculating I is approximately the same (in magnitude) as the relative error in R.

$V = RI \Rightarrow I = V \cdot R^{-1}$ (or $I = \frac{V}{R}$)

$dI = V \cdot (-R^{-2}\, dR)$

So, $dI = -\frac{V}{R^2}\, dR$

Then

$$\frac{dI}{I} = \frac{-\frac{V}{R^2}\, dR}{\frac{V}{R}} = -\frac{V\, dR}{R^2} \cdot \frac{R}{V} = -\frac{dR}{R}$$

So, $\frac{dI}{I} = -\frac{dR}{R}$

(ie: relative error of I & relative error of R are the same in absolute value)