Key

Homework Set 6

Sect 2.6: Implicit Differentiation

1. Consider: $5x^2 - x + xy = 2$

   a. Use implicit differentiation to find $\frac{dy}{dx}$.

   $10x - 1 + y + xy' = 0$

   $xy' = 1 - 10x - y \longrightarrow \frac{dy}{dx} = \frac{1}{x} - 10 - \frac{y}{x}$

   b. Solve the given equation for y. Then find $\frac{dy}{dx}$ without using implicit differentiation.

   $xy = 2 + x - 5x^2$

   $y = \frac{2}{x} + 1 - 5x \longrightarrow \frac{dy}{dx} = -\frac{2}{x^2} - 5$

   c. Verify that the answers for parts (a) and (b) are equivalent.

   $$\frac{dy}{dx} = \frac{1}{x} - 10 - \frac{y}{x} = \frac{1}{x} - 10 - \frac{1}{x}\left(\frac{2}{x} + 1 - 5x\right) = \frac{1}{x} - 10 - \frac{2}{x^2} - \frac{1}{x} + 5$$

   $$= -\frac{2}{x^2} - 5$$

2. Consider: $x = e^y$

   a. Use implicit differentiation to find $\frac{dy}{dx}$.

   $1 = y'e^y \implies \frac{dy}{dx} = e^{-y}$

   b. Solve the given equation for y. Then find $\frac{dy}{dx}$ without using implicit differentiation.

   $y = \ln x \implies \frac{dy}{dx} = \frac{1}{x}$

   c. Verify that the answers for parts (a) and (b) are equivalent.

   $$\frac{dy}{dx} = e^{-y} = e^{-\ln x} = e^{\ln x^{-1}}$$

   $$= x^{-1} = \frac{1}{x}$$

3. Given that $g(x) + x \sin g(x) = x^2$, find $g'(0)$ where $g(0) = \frac{3\pi}{2}$.

   $g'(x) + \sin(g(x)) + x\cos(g(x)) \cdot g'(x) = 2x$

   $g'(0) + \sin\left(\frac{3\pi}{2}\right) + 0 = 0$

   $g'(0) + (-1) = 0$

   $g'(0) = 1$

Use implicit differentiation to compute the derivative $\frac{dy}{dx}$ of the following relations.

4. $x^2 + 2xy - y^2 + x = 2$ when $x = 1$ and $y = 2$

$$2x + 2y + 2xy' - 2yy' + 1 = 0$$
$$2 + 4 + 2y' - 4y' + 1 = 0$$
$$7 - 2y' = 0$$
$$2y' = 7$$
$$\frac{dy}{dx} = 7/2 \quad \text{at } (1,2)$$

5. $x^2 + y^2 = (2x^2 + 2y^2 - x)^2$ when $x = 0$ and $y = \frac{1}{2}$

$$2x + 2yy' = 2(2x^2 + 2y^2 - x)(4x + 4yy' - 1)$$
$$0 + y' = 2(1/2)(0 + 2y' - 1)$$
$$y' = 2y' - 1$$
$$y' = 1 \quad \text{at } (0, 1/2)$$

6. $\ln\left(\frac{x}{y}\right) = x + y$

use: $\ln(x/y) = \ln x - \ln y$

$$\ln x - \ln y = x + y$$
$$\frac{1}{x} - \frac{y'}{y} = 1 + y'$$
$$y - xy' = xy + xyy'$$
$$y - xy = xy' + xyy'$$
$$y - xy = (x + xy)y'$$

So $\frac{dy}{dx} = \frac{y - xy}{x + xy}$

7. $e^{x^2 y} = x + y$

$$(2xy + x^2 y')e^{x^2 y} = 1 + y'$$
$$2xye^{x^2 y} + x^2 y' e^{x^2 y} = 1 + y'$$
$$(x^2 e^{x^2 y} - 1)y' = 1 - 2xye^{x^2 y}$$
$$y' = \frac{1 - 2xye^{x^2 y}}{x^2 e^{x^2 y} - 1}$$

8. $\arctan(xy) = 1 + x^2y$

$$\frac{y + xy'}{(xy)^2+1} = 2xy + x^2y'$$

$$y + xy' = 2x^3y^3 + 2xy + x^2(x^2y^2+1)y'$$

$$y - 2xy - 2x^3y^3 = (x^4y^2 + x^2 - x)y'$$

$$\frac{dy}{dx} = \frac{y - 2xy - 2x^3y^3}{x^4y^2 + x^2 - x}$$

9. $x^{2/3} + y^{2/3} = 4$ when $x = -3\sqrt{3}$ and $y = 1$

$$\tfrac{2}{3}x^{-1/3} + \tfrac{2}{3}y^{-1/3}y' = 0$$

$$y' = -\frac{x^{-1/3}}{y^{-1/3}}$$

$$y' = -\left(\frac{y}{x}\right)^{1/3}$$

$$y' = -\left(\frac{1}{-3\sqrt{3}}\right)^{1/3} = \frac{1}{\sqrt{3}}$$

Use implicit differentiation to compute the 2nd derivative $\frac{d^2y}{dx^2}$ of the following relations. Be sure to write the second derivative only in terms of the x and y.

10. $5x^2 + y^2 = 5$

$$10x + 2yy' = 0$$

$$yy' = -5x$$

$$\left(\text{or } y' = -\frac{5x}{y}\right)$$

then

$$(y')^2 + yy'' = -5$$

$$\left(-\frac{5x}{y}\right)^2 + yy'' = -5$$

$$\frac{25x^2}{y^2} + yy'' = -5$$

$$25x^2 + y^3y'' = -5y^2$$

$$\rightarrow\ y^3y'' = -25x^2 - 5y^2$$

$$y'' = -\frac{25x^2}{y^3} - \frac{5}{y}$$