Key

Homework Set 8

Inverse Functions

(sect 3.2, 3.3, 3.5)

For questions 1 and 2, find the inverse of the given function.

1. $f(x) = 1 + \sqrt{2 + 3x}$

$x = 1 + \sqrt{2+3y}$

$x - 1 = \sqrt{2+3y}$

$(x-1)^2 = 2 + 3y$

$\to (x-1)^2 - 2 = 3y$

$\frac{1}{3}(x-1)^2 - 2/3 = y$

$f^{-1}(x) = \frac{1}{3}(x-1)^2 - 2/3$

2. $y = \frac{e^x}{1+2e^x}$

$x = \frac{e^y}{1+2e^y}$

$x + 2xe^y = e^y$

$(2x-1)e^y = -x$

$e^y = \frac{-x}{2x-1}$

$\to y = \ln\left(\frac{-x}{2x-1}\right)$

$f^{-1}(x) = \ln\left(\frac{x}{1-2x}\right)$

3. Suppose f^{-1} is the inverse function of a differentiable function f and $f(4) = 5$, $f'(4) = \frac{2}{3}$. Find $(f^{-1})'(5)$.

rule: $(f^{-1})'(5) = \frac{1}{f'(f^{-1}(5))} = \frac{1}{f'(4)} = \frac{1}{2/3} = 3/2$

4. Suppose f^{-1} is the inverse function of a differentiable function f and let $G(x) = \frac{1}{f^{-1}(x)}$. If $f(3) = 2$ and $f'(3) = \frac{1}{9}$, find $G'(2)$.

$G'(x) = \frac{d}{dx}\left[\frac{1}{f^{-1}(x)}\right] = \frac{d}{dx}\left[f^{-1}(x)\right]^{-1} = -\left[f^{-1}(x)\right]^{-2}(f^{-1})'(x)$

$G'(2) = -\frac{(f^{-1})'(2)}{(f^{-1}(2))^2} = -\frac{\frac{1}{f'(f^{-1}(2))}}{3^2} = -\frac{\frac{1}{f'(3)}}{9} = -\frac{\left(\frac{1}{1/9}\right)}{9} = -1$

5. Let $f(x) = 2x^3 + 3x^2 + 7x + 4$. Compute $(f^{-1})'(4)$.

$(f^{-1})'(4) = \frac{1}{f'(f^{-1}(4))} = \frac{1}{f'(0)}$

$= \frac{1}{7}$

$f(x) = 4 \Rightarrow x = 0 \Rightarrow f^{-1}(4) = 0$

$f'(x) = 6x^2 + 6x + 7$

6. Let $f(x) = x^3 + 3\sin x + 2\cos x$. Compute $(f^{-1})'(2)$.

$(f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(0)} = \frac{1}{3}$

$f(x) = 2 \Rightarrow x = 0 \Rightarrow f^{-1}(2) = 0$

$f'(x) = 3x^2 + 3\cos x - 2\sin x$

$f'(0) = 3$

7. $f(x) = 9 - x^2$ where $0 \le x \le 3$

a. Use the rule for the derivative of an inverse function to find $(f^{-1})'(8)$.

$$(f^{-1})'(8) = \frac{1}{f'(f^{-1}(8))} = \frac{1}{f'(1)} = -\frac{1}{2}$$

$f(x) = 8 \Rightarrow x = 1 \Rightarrow f^{-1}(8) = 1$

$f'(x) = -2x$

$f'(1) = -2$

b. Calculate $f^{-1}(x)$.

$x = 9 - y^2$

$y^2 = 9 - x$

$y = \sqrt{9-x} \implies f^{-1}(x) = \sqrt{9-x}$

c. Compute $(f^{-1})'(8)$ from the inverse function found in part (b).

$$(f^{-1})'(x) = \frac{-1}{2\sqrt{9-x}} \implies (f^{-1})'(8) = \frac{-1}{2\sqrt{1}} = -\frac{1}{2}$$

8. $f(x) = \frac{1}{x-1}$ where $x > 1$

a. Use the rule for the derivative of an inverse function to find $(f^{-1})'(2)$.

$$(f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(3/2)} = \frac{1}{\frac{-1}{(1/2)^2}} = -\frac{1}{4}$$

$f(x) = 2 \Rightarrow x = 3/2 \Rightarrow f^{-1}(2) = \frac{3}{2}$

$f'(x) = \frac{-1}{(x-1)^2}$

b. Calculate $f^{-1}(x)$.

$x = \frac{1}{y-1}$

$xy - x = 1$

$xy = x + 1$

$y = 1 + 1/x$

$\rightarrow f^{-1}(x) = 1 + 1/x$

c. Compute $(f^{-1})'(2)$ from the inverse function found in part (b).

$(f^{-1})'(x) = -1/x^2$

$(f^{-1})'(2) = -1/4$

Use trig identities, implicit differentiation, and inverse functions to show the following derivative rules.

9. $\frac{d}{dx}(\operatorname{arccot} x) = -\frac{1}{x^2+1}$

$y = \operatorname{arccot} x$

$\cot y = x$

$\rightarrow \frac{d}{dx}[\cot y] = 1$

$-\csc^2 y \cdot \frac{dy}{dx} = 1$

$\rightarrow \frac{dy}{dx} = -\frac{1}{\csc^2 y}$

$\frac{dy}{dx} = -\frac{1}{1+x^2}$

$\sin^2 y + \cos^2 y = 1$

$1 + \cot^2 y = \csc^2 y$

$1 + x^2 = \csc^2 y$

10. $\frac{d}{dx}(\operatorname{arcsec} x) = \frac{1}{x\sqrt{x^2-1}}$

$y = \operatorname{arcsec} x$

$\sec y = x$

$\frac{d}{dx}[\sec y] = 1$

$\rightarrow \sec y \tan y \cdot \frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{\sec y \tan y} = \frac{1}{x\sqrt{x^2-1}}$

$\sin^2 y + \cos^2 y = 1$

$\tan^2 y + 1 = \sec^2 y$

$\tan^2 y = \sec^2 y - 1$

$\tan^2 y = x^2 - 1$

$\tan y = \sqrt{x^2-1}$