Project 3

The Origins of Calculus

(sect 2.1 – 2.2, 2.6, 2.8)

I. Background

Mathematics has a long history. Some of the earliest recorded uses of mathematics include the ancient Egyptians using math (geometry) to build their pyramids and the Babylonians' method for computing a square root. The ancient Greeks improved on geometry and gave us the notion of a proof. The ancient Chinese gave us the Chinese remainder theorem (they also independently developed all the same math as in the west). The ancient Indians gave us algebra.

But for all the math that was known and used in business, architecture, finance, science, etc., standardized mathematical symbols and formulas were not developed until the Middle Ages. For example, in the 1500s, it was known how to solve quadratic, cubic, and quartic equations, but there were no formulas. Instead, there were just lists of steps to follow.

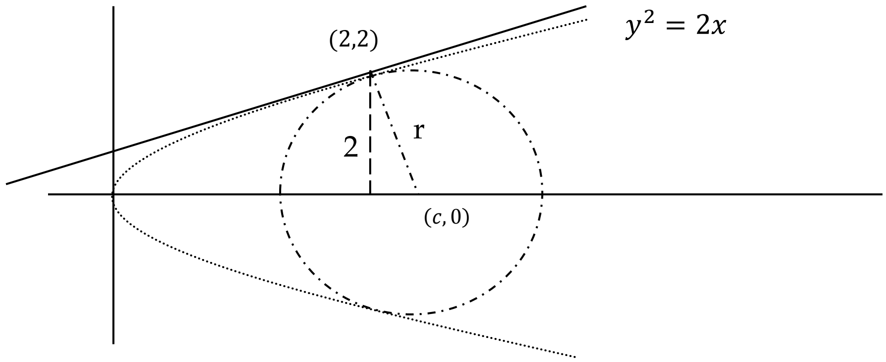
Science was improving. More people were looking at the known universe and trying to figure out what was going on based off their observations. Nicolaus Copernicus (1473 – 1543) proposed that the sun was at the center of the universe and backed up his theory with observations and trig calculations. Then Galileo Galilei (1564 – 1642) expanded on Copernicus's theory. Galileo also worked on experiments with gravity and constructed telescopes which could see farther into space than earlier telescopes. While new theories in science were occurring rapidly, many of the findings could not be proven because the mathematics needed to prove these hypotheses had yet to be created.

Two main people are credited as creating calculus: Sir Isaac Newton and Gottfried Leibnitz. While Leibnitz published his calculus first, it is thought that Newton developed his calculus a few years before Leibnitz. This created a large controversy between English and continental mathematicians at the time. Today mathematicians mainly use the notation of Leibnitz since his notation is more versatile then Newton's. However, in problems where time is one of several variables used, people often use Newton's notion (to differentiate from having a variable in time versus a variable in distance).

II. Rene Descartes (1596 - 1650)

Descartes is known as the father of modern philosophy and is the founder of analytic geometry. He was the first to plot equations on a graph, coming up with the idea of the x-y plane to make dealing with curves easier. This is why the x-y plane is officially called the Cartesian Coordinate Plane. Because of Descartes, calculus was able to be developed within a generation. Descartes used geometry (since calculus had yet to be developed) to find both the equation of the tangent line and the slope of the line tangent to a curve. He often worked with sideways parabolas.

Let's walk through what Descartes would do if he wanted to calculate the slope of the line tangent to $y^2 = ax$ at (a, a) where a is some positive constant. We'll assume that $a = 2$.



We start by considering a circle tangent to $y^2 = 2x$ at (2,2) and whose center is on the x-axis at $(c, 0)$. Let r be the radius of this circle. Then the equation of this circle is

Equation of a circle: $(x - c)^2 + y^2 = r^2$ with center at $(c, 0)$ and radius r

distance between (2, 2) & $(c, 0)$: $r = \sqrt{(2 - c)^2 + (2 - 0)^2} \quad \Longrightarrow \quad r^2 = (2 - c)^2 + 2^2$

Use the equation of the circle to get rid of r; then use the equation of the line to get rid of y^2

$$\Longrightarrow \qquad (x - c)^2 + y^2 = (2 - c)^2 + 2^2$$

$$\Longrightarrow \qquad (x - c)^2 + \left(\qquad \right) = (2 - c)^2 + 2^2$$

Next we need to simplify the expression. Start by squaring, and then combining like terms.

$$\Longrightarrow \qquad x^2 - 2xc + c^2 + (\qquad) = 2^2 - 2 \cdot 2c + (\qquad) + 2^2$$

$$\Longrightarrow \qquad x^2 - 2xc + 2x = 2 \cdot 2^2 - 2 \cdot 2c$$

$$\Longrightarrow \qquad x^2 - 2xc + 2x - 8 + 4c = 0$$

$$\Longrightarrow \qquad x^2 + (2 - 2c)x + (4c - 8) = 0$$

The way the circle is set up there will be exactly 1 solution to this quadratic equation, and a quadratic equation with one solution must be able to be written in the form: $(x + d)^2 = 0$. In the next step, we will be trying to find what "d" is. To do this, we complete the square on

$$x^2 + (2 - 2c)x \quad \text{to get:} \quad x^2 + (2 - 2c)x + \left(\frac{2-2c}{2}\right)^2$$

Which means that $x^2 + (2 - 2c)x + \left[4c - \left(\qquad \right)\right] = x^2 + (2 - 2c)x + \left(\frac{2-2c}{2}\right)^2$

And cancelling like terms yields:

$$\Longrightarrow \qquad (4c - 8) = \left(\qquad \qquad \right)$$

$$\Rightarrow \quad 16c - 32 = 4 - 8c + 4c^2$$

$$\Rightarrow \quad 0 = 4c^2 - 24c + (\qquad)$$

$$\Rightarrow \quad 0 = (2c - 6)^2$$

$$\Rightarrow \quad 2c - 6 = 0$$

$$\Rightarrow \quad c =$$

Thus, we need to choose the point $\left(\frac{3a}{2}, 0\right) = (\quad$,0) as the center of the circle. This circle will then automatically be tangent to the line $y^2 = ax$ at the point (a, a). So the slope of the tangent line at (a, a) will be perpendicular to the slope between the points (a, a) and $\left(\frac{3a}{2}, 0\right)$.

Find the slope between the points (a, a) and $\left(\frac{3a}{2}, 0\right)$, where a is left as a:

What is the slope of the tangent line at (a, a) in terms of a?

Use implicit differentiation to find y' of the equation $y^2 = ax$:

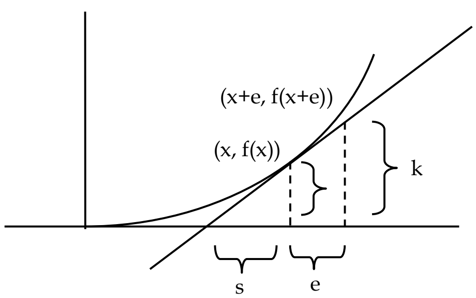
What is y' at the point (a, a)? Does your answer agree with the answer from Descartes' method?

Why would you hypothesize that Descartes' method never really caught on?

III. Pierre de Fermat (1601 – 1665)

Fermat spent most of his life in Toulouse, France where he worked as a lawyer. (At this time, France had a number of famous and influential mathematicians. The main three were Blasé Pascal, Rene Descartes, and Pierre de Fermat.) Fermat worked on math problems as a hobby. His main contributions to mathematics were in number theory where he worked on ideas such as the properties of prime numbers. He is probably most known for his famous Last Theorem[1]. Fermat also worked with Pascal on "the problem of the points" which led to advances in probability theory. Additionally, both Fermat and Descartes worked in the field of analytical geometry (applies algebra techniques to geometry). However, they did not work together, and in fact, did not like each other.

Fermat's method for finding lines tangent to a curve at a point has components that are almost identical to the method we use today to find derivatives even though he did not use limits. Let's walk through what Fermat would do if he wanted to find the slope of a line tangent to $f(x) = x^2$ at the general point x.



We pick an arbitrary point close to x: $(x+e, f(x+e))$ where e is small

Then $k \approx f(x+e)$

After drawing the diagram and labeling the points, we use similar triangles to get:

$$\frac{s+e}{\Box} = \frac{k}{f(x)}$$

Next we solve for s:

$$1 + \frac{\Box}{s} = \frac{k}{f(x)} \quad \Longrightarrow \quad \frac{\Box}{s} = \frac{k - \Box}{f(x)} \quad \Longrightarrow \quad s = \frac{e \cdot f(x)}{k - \Box}$$

Recall that $k = f(x+e)$, so if we plug in $f(x+e)$ for k, we get:

$$s = \frac{e \cdot f(x)}{f(x+e) - \Box} \quad \Longrightarrow \quad s = \frac{\Box}{\left(\frac{f(x+e) - f(x)}{e}\right)}$$

But $f(x) = x^2$, so if we plug in x^2 for $f(x)$, we get:

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[1] Fermat's Last Theorem:
For $n > 2$, there are no non-zero integers a, b, and c such that $a^n + b^n = c^n$

$$s = \frac{x^2}{\left(\dfrac{\boxed{}}{e}\right)} = \frac{x^2}{\boxed{}} = \left\{\begin{matrix}\text{now ignore} \\ \text{any term} \\ \text{with "e" in it}\end{matrix}\right\} = \frac{x^2}{\boxed{}} = \boxed{}$$

Then using the expression found for s above, the <u>slope</u> of the tangent line at $(x, f(x))$ is given by

$$\frac{f(x)}{s} = \frac{\boxed{}}{\boxed{}} = \boxed{}$$

Is this the answer you expected? ____________ Why or why not? _________________________

IV. Sir Isaac Newton (1642 – 1727)

Englishman Sir Isaac Newton did not publish works unless he was forced since he did not want to deal with criticism; however, he would present his work at lectures and to the British Royal Society (Britain's national science society) of which he was the head for a number of years. Newton used his calculus to prove his theories of the universe. He published his theories in 1687 in 3 volumes, which he called the *Philosophiae Natualis Principia Mathematica*. It is now referred to as *The Principia*. In this set, he tried to explain the motions of the heavenly bodies. Among other things, he introduced his 3 Laws of Motion in this work. Now, while he had used calculus to develop and prove his theories and laws in this work, he knew that the average person would not be able to understand both the new theories in science and the new math he had developed, so when he wrote the book(s) he left out the calculus.

When Newton worked with calculus there were 2 central ideas that he considered:

1. Given the length of the space at any time, find the speed of the motion at the time progressed. (ie: find the velocity of an object – 1st derivative)
2. Given the speed of the motion at any time, find the length of the space described at any time proposed. (ie: given the velocity, find the position – antiderivative)

We'll only consider the first idea. The terms he used were different than the ones we've been using:

- Fluent – the quantity x, which is changing over time. This is the function.
- Fluxion – the speed at which x is changing, called $\dot{x}$ (ie: first derivative with respect to time).
- Moment – the amount by which the fluent changes over an infinitely small period of time. σ is the infinitely small period of time. The moment is $\dot{x}\sigma$.

Let's walk through what Newton would do if he wanted to calculate the derivative of $y = x^2 + 2x - 5$. This process is called the Method of Fluxions since both x and y are fluents.

Step 1: replace y with $(y + \dot{y}\sigma)$ and x with $(x + \dot{x}\sigma)$

$$(y + \dot{y}\sigma) = (\qquad\qquad)^2 + 2(\qquad\qquad) - 5$$

Step 2: simplify the expression

$$(y + \dot{y}\sigma) = x^2 + (\qquad\qquad) + (\dot{x}\sigma)^2 + (\qquad\qquad) + 2\dot{x}\sigma - 5$$

Substitute $y = x^2 + 2x - 5$ in for y

$$(\qquad\qquad) + \dot{y}\sigma = x^2 + (\qquad\qquad) + (\dot{x}\sigma)^2 + 2x + 2\dot{x}\sigma - 5$$

Then cancel terms to get:

$$\dot{y}\sigma = (\qquad\qquad) + (\dot{x}\sigma)^2 + 2\dot{x}\sigma$$

Step 3: solve for $\dot{y}$

$$\dot{y} =$$

$\dot{x}\sigma$ is infinitesimally small, so we "cast it out." Remove all terms with σ still in them:

$$\dot{y} =$$

Step 4: solve for $\dot{y}/\dot{x}$

$$\frac{\dot{y}}{\dot{x}} =$$

Find the derivative of $y = x^2 + 2x - 5$ the way our textbook shows:

$\frac{dy}{dx} =$ [ ] is this the same answer as $\dot{y}/\dot{x}$? ____________

V. Gottfried Wilhelm Leibnitz (1646 – 1716)

Born in Leipzig, Germany, Leibnitz was quite a scholar. By the time he was twenty, he applied to law school, and had all the qualifications, but he was turned down. The school thought he was too young. In 1667, he received a political post in Mintz, Germany. (Recall, there were many wars centered around Germany.) As part of his duties, he was sent to France in 1672 to try to get France not to march on Germany. While he was in Paris, he met several prominent mathematicians.

During the 4 years Leibnitz was in France, he studied mathematics on the side. He worked with finite sums (ie: adding a finite number of numbers) and infinite sums (ie: adding up a list of numbers that doesn't end). Through this work, he developed the concept of integrals (the topic covered in a Calculus 2 course). In the course of this work, he by necessity discovered a number of derivative rules. We'll look at 3 of his main derivative rules below. Be aware, when Leibnitz uses x or y, he is assuming that x and y are functions. Notice also that instead of using derivatives he is in fact using differentials here:

$$d(x \cdot y) = y \cdot dx + x \cdot dy$$

$$d\left(\frac{x}{y}\right) = \frac{y \cdot dx - x \cdot dy}{y^2}$$

$$d(x^n) = n \cdot x^{n-1} \cdot dx$$

Use Leibnitz's rules to compute the following differentials:

1. $d(t \cdot e^t) =$

$x =$	$dx =$
$y =$	$dy =$

2. $d\left(\frac{\cos t}{t^3}\right) =$

$x =$	$dx =$
$y =$	$dy =$

3. $d(x^6) =$

VI. References and More Information

a. Books

i. Carl Boyer & Uta Merzbach, *A History of Mathematics* (New York: Wiley, 1989).
ii. C.H. Edwards, *The Historical Development of the Calculus* (New York: Springer-Verlag, 1979).
iii. Howard Eves, *An Introduction to the History of Mathematics*, 6th ed. (New York: Saunders, 1990).
iv. Victor Katz, *A History of Mathematics* (Boston: Pearson, 2004).
v. Morris Kline, *Mathematical Thought from Ancient to Modern Times* (New York: Oxford University Press, 1972).

b. Websites

i. http://royalsociety.org/history-of-science/
ii. http://www-history.mcs.st-and.ac.uk/BiogIndex.html
iii. http://library.thinkquest.org/22584/