Sec 1.1 and 1.2 Functions: Review of Algebra and Trigonometry

A. Functions and Relations

DEFN **Relation**: A set of ordered pairs.
(x,y) $\rightarrow$ (domain, range)

DEFN **Function**: A correspondence from one set (the domain) to anther set (the range) such that each element in the domain corresponds to exactly one element in the range.

Example: Determine whether each of the following is an example of a function or not.

1.) $\{(1,1)(3,2)(5,3)\}$

2.) $\{(1,1)(2,4)(3,-5)(2,-4)\}$

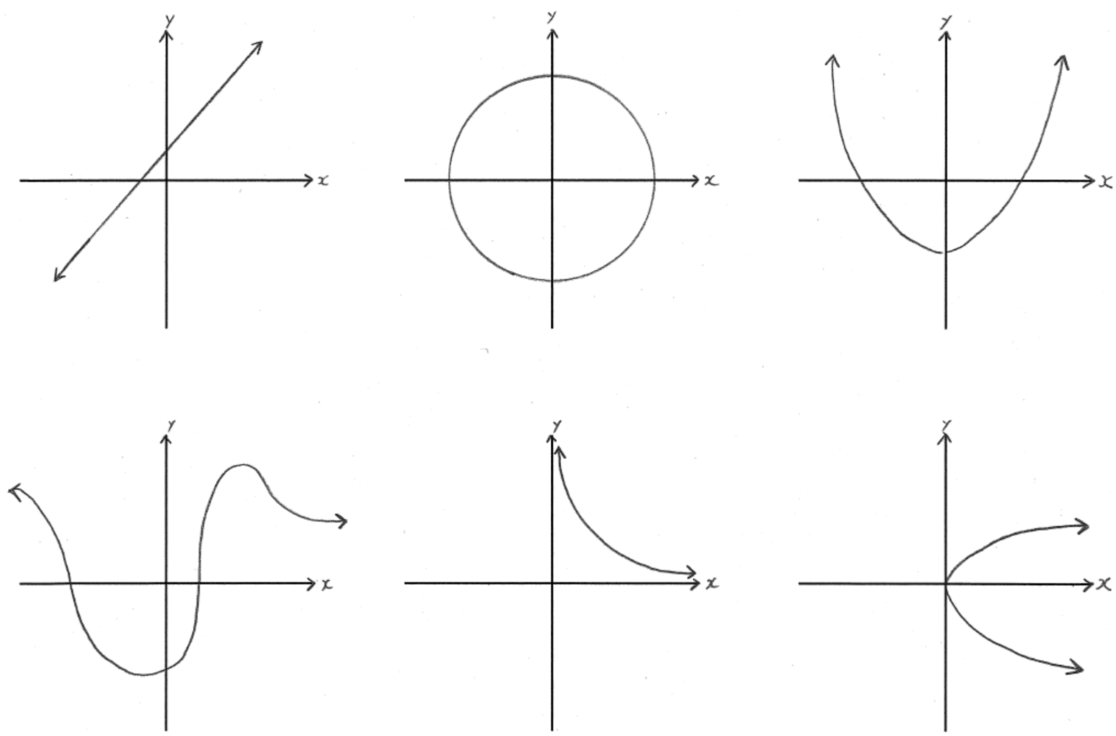
3.) $x^2+2y=0$

4.) $x^2+y^2=1$

Vertical Line Test for Functions

Vertical Line Test for Functions: If any vertical line intersects a graph more than once, then the graph is **not** a function.

Example: Determine whether each of the following is a function or not by the Vertical Line Test.



B. Domain and Range of a Function

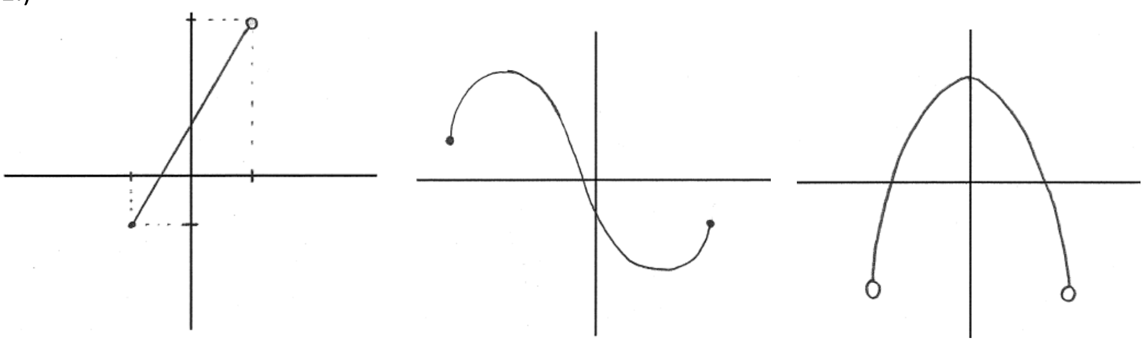
DEFN **Domain**: Input $\longrightarrow$ x-values (i.e. All of the values of x that I may plug into a function.)

DEFN **Range**: Output $\longrightarrow$ y-values (i.e. All of the values of y that a function can attain)

Function Notation: $f(x) = y$

Example: Give the domain and range (in interval notation) for each of the following

1.)



Domain:_________________ Domain:_________________ Domain:_________________

Range: _________________ Range: _________________ Range: _________________

Example: Give the domain for each of the following (in interval notation and as an inequality)

2.) $g(x) = \sqrt{2-x}$

3.) $f(x) = \dfrac{x^2 - 4}{x+2}$

4.) $g(x) = \dfrac{3x}{\sqrt{x+5}}$

5.) $h(x) = \ln(6x-3)$

* The 3 functions for which we will most frequently have domain restrictions (in this course) are: **fractions (aka…rational functions)**, **radicals** and **logarithms**.

C. Linear Models

Definition

$$\textbf{Slope} = m = \frac{rise}{run} = \frac{\Delta\, y}{\Delta\, x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Slope	$m = \frac{y_2 - y_1}{x_2 - x_1}$	(Use when given two points to find slope)
Slope-Intercept Form	$y = mx + b$	(Use when given slope and y - intercept)
Point-Slope Form	$(y - y_1) = m \cdot (x - x_1)$	(Use when given one point and slope)
General Form	$A \cdot x + B \cdot y + C = 0$	
Horizontal Line	$y = b$ (where b = constant)	
Vertical Line	$x = c$ (where c = constant)	

Parallel Lines

Two lines are parallel if and only if they have the same slope.

For two lines $y_1 = m_1 x + b_1$ and $y_2 = m_2 x + b_2$ we have $y_1 \parallel y_2 \quad \Leftrightarrow \quad m_1 = m_2$

Perpendicular lines

Two lines are perpendicular if and only if the product of their slope = -1.

For two lines $y_1 = m_1 x + b_1$ and $y_2 = m_2 x + b_2$ we have $y_1 \perp y_2 \quad \Leftrightarrow \quad m_1 = -\frac{1}{m_2}$

Examples: Find the equation of the line:

1.) that passes through point (0 , -3) with slope = -2

2.) that passes through points (3 , -2) and (4 , 5)

3.) that passes through point (0, 0) and is parallel to the line $y = 3x - 9$

4.) that passes through point (2 , -4) and is perpendicular to the line $y = \frac{1}{2}x + 3$

5.) Find the slope and y-intercept of the line $9x - 3y - 3 = 0$

D. Classes of Functions

1. Power Functions

For any real number m, a function in the form $f(x) = x^m$ is called a Power Function

2. Polynomials

Definition

A *polynomial function* is a function in the form $f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$ where $a_0 \neq 0$ and n is a positive integer.

Examples: State whether each is a polynomial:

1.) $g(x) = x^2 + 5x + 6$

2.) $f(x) = x^3 + 7x - \sqrt{x}$

3.) $f(x) = \frac{x^2}{3} + 8x$

4.) $h(x) = x^{\frac{2}{3}} - x + 5$

5.) $f(x) = \frac{4}{x^2} + 6x - 1$

3. Rational Functions

A Rational Function is the quotient of two polynomial functions:

A Rational Function is a function of the form $f(x) = \dfrac{p(x)}{q(x)} = \dfrac{a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x \ + a_0}{b_m x^m + b_{m-1}x^{m-1} + ... + b_1 x \ + b_0}$

Asymptotes

An asymptote is an imaginary line that the graph of a function approaches as the function approaches a restricted number in the domain or as it approaches infinity.

I. Locating Vertical Asymptotes

If $f(x) = \dfrac{p(x)}{q(x)}$ is a rational function, $p(x)$ and $q(x)$ have no common factors and n is a zero of $q(x)$, then the line $x = n$ is a vertical asymptote of the graph of $f(x)$.

II. Locating Horizontal Asymptotes

Let $f(x) = \dfrac{p(x)}{q(x)} = \dfrac{a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x \ + a_0}{b_m x^m + b_{m-1}x^{m-1} + ... + b_1 x \ + b_0}$

i. If n < m, then $y = 0$ is the horizontal asymptote ("Bottom Heavy")

ii. If n = m, then the line $y = \dfrac{a_n}{b_m}$ is the horizontal asymptote ("Equal Degree")

iii. If n > m, there is NO horizontal asymptote. (But there will be a slant/oblique asymptote) ("Top Heavy")

Examples: Find all vertical and horizontal asymptotes:

1.) $f(x) = \dfrac{15x}{3x^2 + 1}$

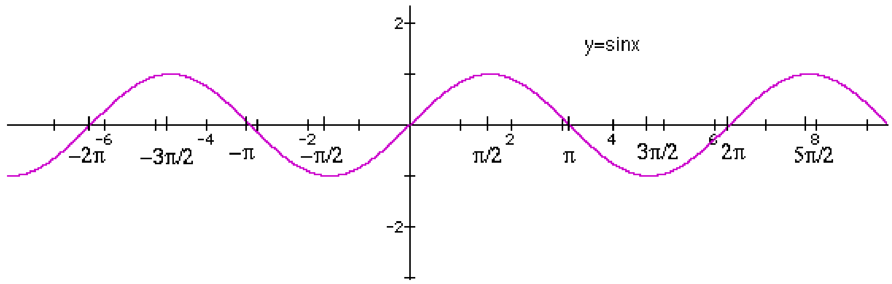
2.) $g(x) = \dfrac{15x^3}{3x^2 + 1}$

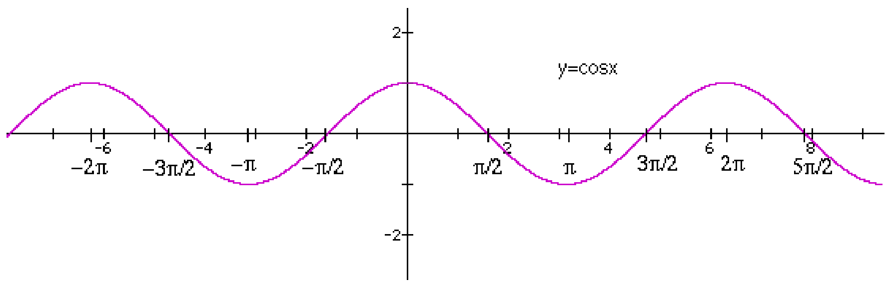
3.) $h(x) = \dfrac{-3x + 7}{5x - 2}$

4.) $k(x) = \dfrac{2x - x^2}{x^2 - 2x - 3}$

4. Trigonometric Functions

sin(x)	csc(x)
cos(x)	sec(x)
tan(x)	cot(x)





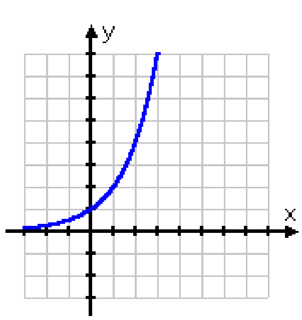
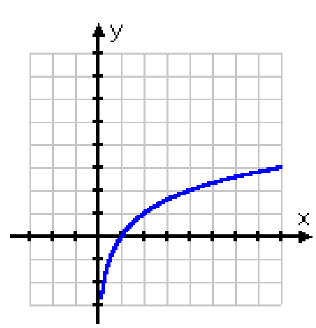
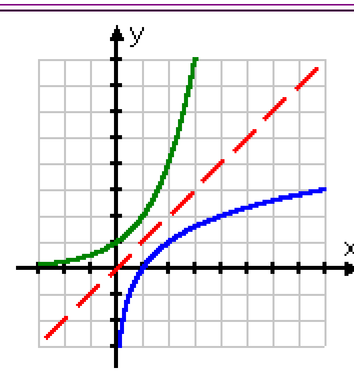
5. Exponential and Logarithmic Functions

DEFN: An exponential function is a function in the form $f(x) = a^x$. (i.e. the variable x is in the exponent)

DEFN: A logarithmic function is a function in the form $f(x) = \log_a x$. (i.e. the variable x is in the expression)

$y = \log_b x$ "**y** is equal to log base **b** of **x**" - Here "**b**" is the BASE NUMBER and "**x**" is the VARIABLE.

$\log_b x = y$ means exactly the same thing as $b^y = x$

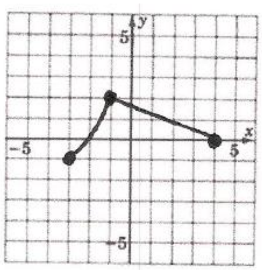
$y = 2^x$	$y = \log_2(x)$	**Comparison of the two graphs, showing the inversion line in red.**
		

E. Transformations of Functions

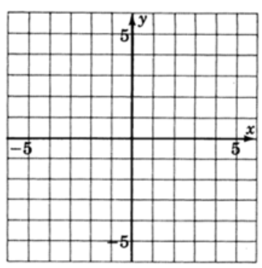
Vertical Shifts	$f(x)+C$	↑	Moves Graph **UP C** units
	$f(x)-C$	↓	Moves Graph **DOWN C** units
Horizontal Shifts	$f(x-C)$	→	Moves Graph RIGHT C units
	$f(x+C)$	←	Moves Graph LEFT C units
Vertical and Horizontal Reflections	$-f(x)$	↕	Flips Graph About x-axis
	$f(-x)$	↔	Flips Graph About y-axis
Vertical Stretching/ Compressing	$c \cdot f(x)$ for $c > 1$	↑ ↓	Graph Vertically Stretches by a Factor of C
	$c \cdot f(x)$ for $0 < c < 1$	↓ ↑	Graph Vertically Shrinks by a Factor of C

Example: Use the given graph of ***f(x)*** to sketch each of the following.

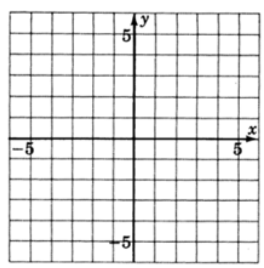
f(x)



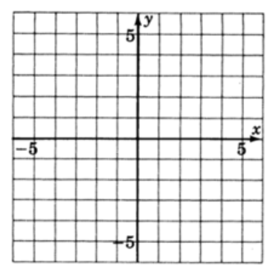
a. f(x) + 2



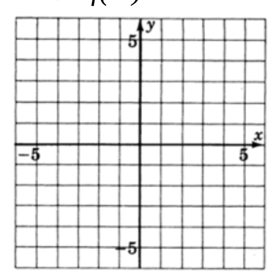
b. f(x + 2)



c. –f(x)



d. 2f(-x)

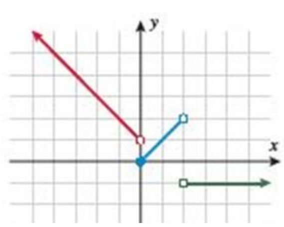


F. Combinations of Functions

1. Piecewise-Defined Functions

A Piecewise Function is a function that has specific (and different) definitions on specific intervals of x.

$$f(x)=\begin{cases} -x+1 & x<0 \\ x & 0 \le x < 2 \\ -1 & x > 2 \end{cases}$$



Domain: ______________________________

Range: ______________________________

2. Sums, Differences, Products and Quotients of Functions

Sum	$(f+g)(x)=f(x)+g(x)$
Difference	$(f-g)(x)=f(x)-g(x)$
Product	$(f\cdot g)(x)=f(x)\cdot g(x)$
Quotient	$\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}$

Example

1. If $f(x) = 2x+1$ *and* $g(x) = x^2 - 3$ find each of the following.

a. $f(4)$ *b.* $g(2x)$ *c.* $f(3x-4)$ *d.* $f(x)+g(x)$ *e.* $f(x)g(x)$

3. Composition of Functions

Notation	$(f\circ g)(x)=f(g(x))$

Example:

1.) For the functions $f(x)=\sqrt{x}$ and $g(x)=x+2$ find

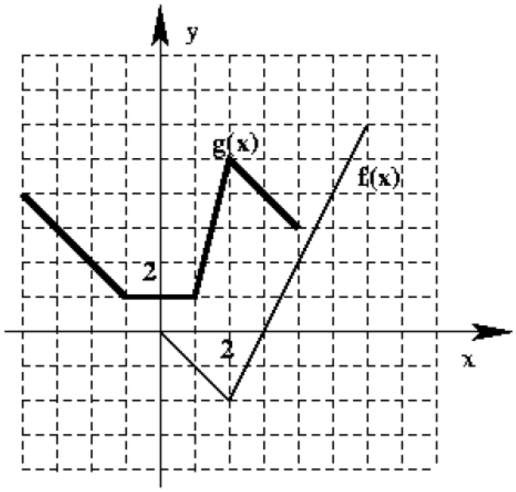
a.) $(f\circ g)(x)=$

b.) $(g\circ f)(x)=$

c.) $(f\circ g)(2)=$

d.) $(g\circ g)(x)=$

Example: For the functions f(x) and g(x) given in the graph find



a.) $(f \circ g)(2) =$

b.) $(f \circ g)(3) =$

c.) $(g \circ f)(2) =$

d.) $(g \circ f)(3) =$

e.) $(f \circ f)(4) =$

f.) $(g \circ g)(4) =$

G. Symmetry

Symmetry: **Even** functions… f(x) = f(- x) …. Symmetric about the y-axis …. If (a,b) then (-a,b) **Odd** functions…f(- x) = - f(x) …. Symmetric about the origin …. If (a,b) then (-a,-b)

1. State whether the following functions are even, odd, or neither.

a. $f(x) = x^5 + 5x$

b. $f(x) = 1 - x^4$

c. $f(x) = 2x - x^2$

d. $f(x) = 2\sin(x)$

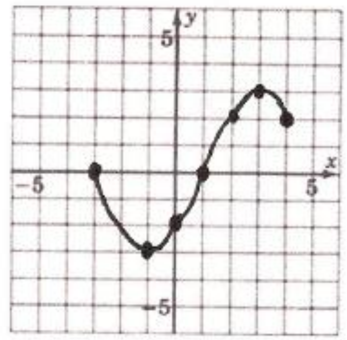
e. $f(x) = \cos(x) - 1$

f. $f(x) = |x| - 3$

H. Function Properties

-	**Increasing** functions rise from left to right	**Decreasing** functions fall from left to right
	Positive functions are above the x-axis	**Negative** functions are below the x-axis
	For all of these above, you use the x-values to state your answers!	

1. Find each of the following using the given function.



a. $f(x) > 0$

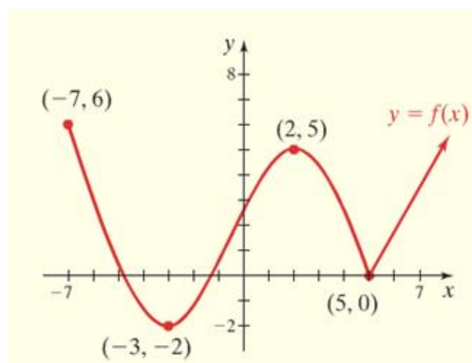
b. $f(x) \leq 0$

c. *increasing*

d. *decreasing*

e. *domain and range*

2. Find each of the following using the given function.



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Sec 1.2 Review of Algebra and Trigonometry

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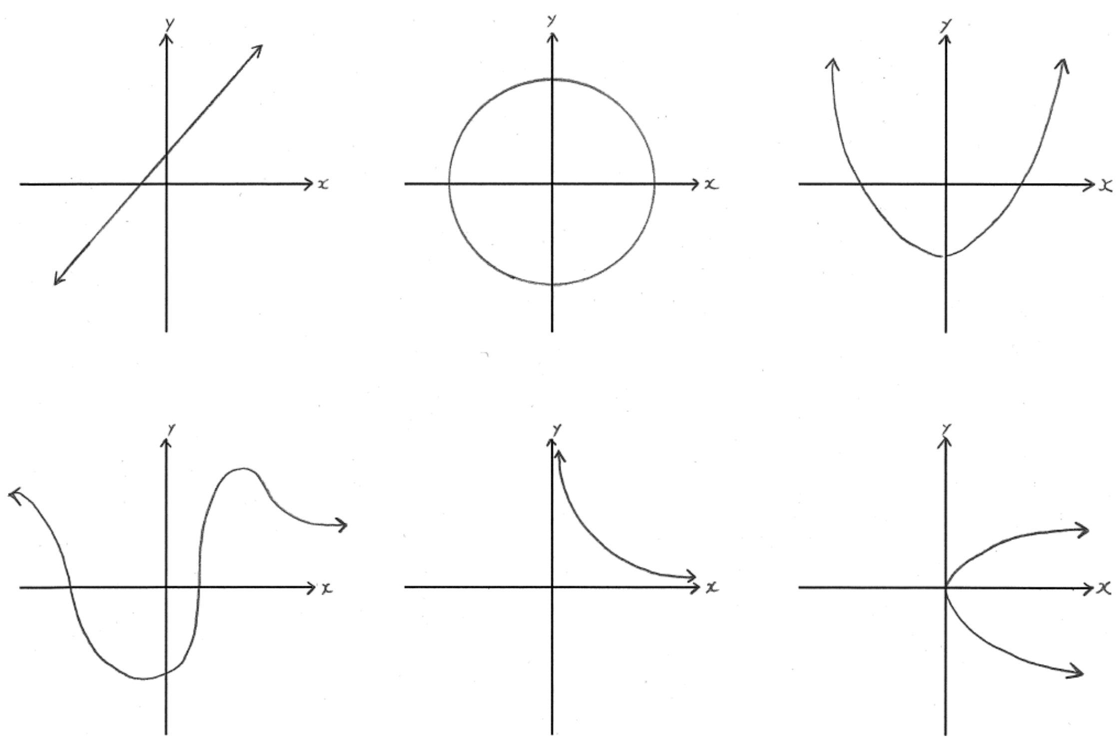

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Example: Give the domain and range (in interval notation) for each of the following

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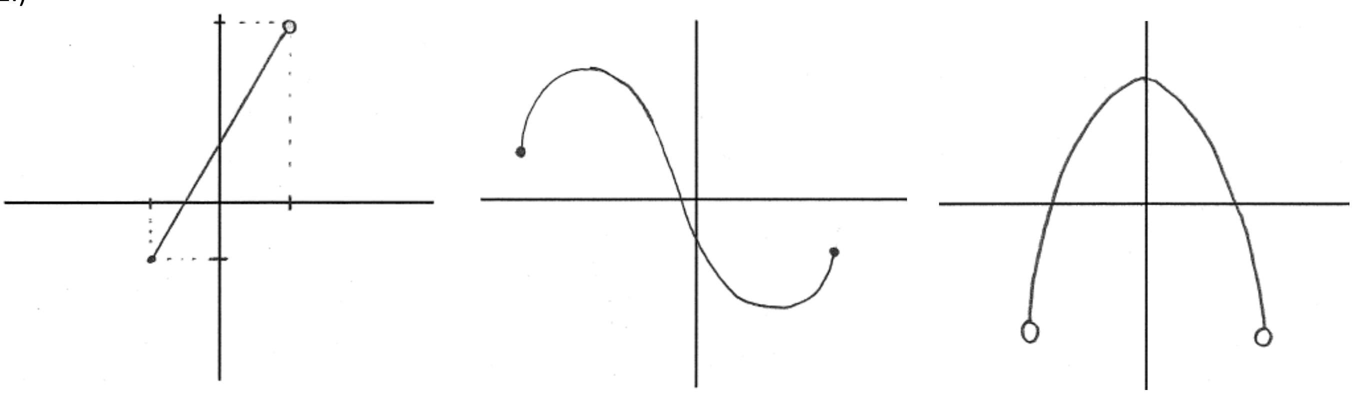

Domain:_________________ Domain:_________________ Domain:_________________

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Definition

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General Form	$A \cdot x + B \cdot y + C = 0$	
Horizontal Line	$y = b$ (where b = constant)	
Vertical Line	$x = c$ (where c = constant)	

Parallel Lines

Two lines are parallel if and only if they have the same slope.

For two lines $y_1 = m_1 x + b_1$ and $y_2 = m_2 x + b_2$ we have $y_1 \parallel y_2 \iff m_1 = m_2$

Perpendicular lines

Two lines are perpendicular if and only if the product of their slope = -1.

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A Rational Function is the quotient of two polynomial functions:

A Rational Function is a function of the form $f(x) = \dfrac{p(x)}{q(x)} = \dfrac{a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x \;\; + a_0}{b_m x^m + b_{m-1}x^{m-1} + ... + b_1 x \;\; + b_0}$

<u>*Asymptotes*</u>

An asymptote is an imaginary line that the graph of a function approaches as the function approaches a restricted number in the domain or as it approaches infinity.

I. Locating Vertical Asymptotes

If $f(x) = \dfrac{p(x)}{q(x)}$ is a rational function, $p(x)$ and $q(x)$ <u>have no common factors</u> and n is a zero of $q(x)$, then the line $x = n$ is a vertical asymptote of the graph of $f(x)$.

II. Locating Horizontal Asymptotes

Let $f(x) = \dfrac{p(x)}{q(x)} = \dfrac{a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x \;\; + a_0}{b_m x^m + b_{m-1}x^{m-1} + ... + b_1 x \;\; + b_0}$

i. If n < m, then $y = 0$ is the horizontal asymptote

ii. If n = m, then the line $y = \dfrac{a_n}{b_m}$ is the horizontal asymptote

iii. If n > m, there is NO horizontal asymptote. (But there will be a slant/oblique asymptote.)

Examples: Find all asymptotes:

1.) $f(x) = \dfrac{15x}{3x^2 + 1}$

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3.) $h(x) = \dfrac{-3x + 7}{5x - 2}$

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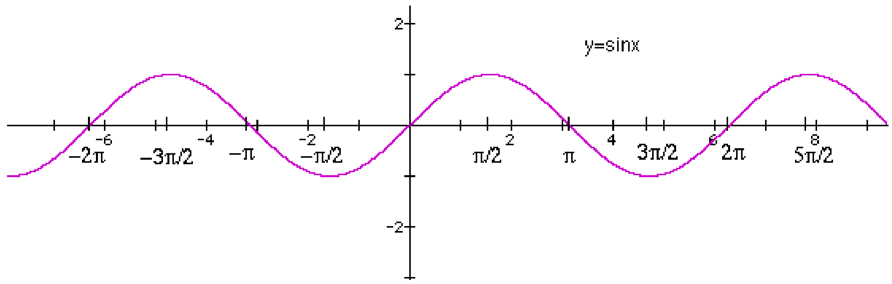

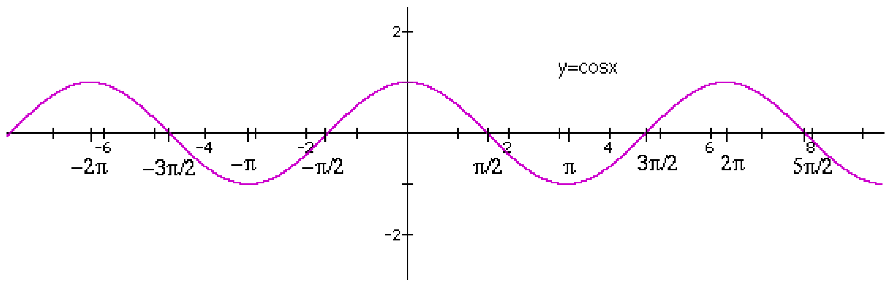

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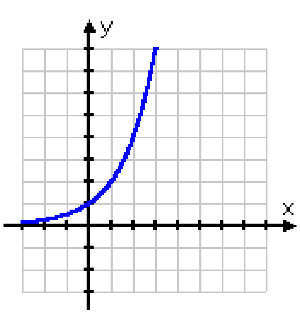	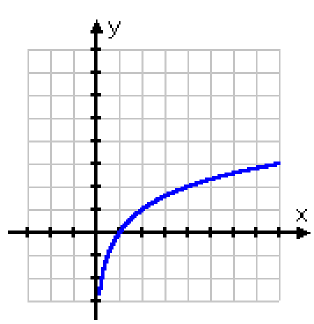	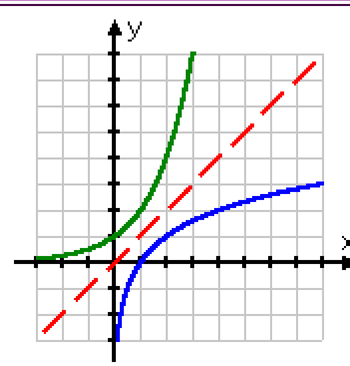

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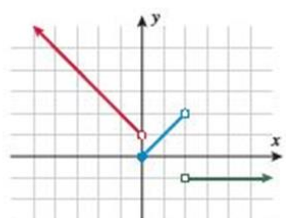


2. Sums, Differences, Products and Quotients of Functions

Sum	$(f+g)(x)=f(x)+g(x)$
Difference	$(f-g)(x)=f(x)-g(x)$
Product	$(f\cdot g)(x)=f(x)\cdot g(x)$
Quotient	$\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}$

3. Composition of Functions

$f(x) = x^2+5x+2$	$f(☺) = (☺)^2+5(☺)+2$
$g(x)=\dfrac{2x}{\sqrt{x+1}}$	$g(\quad)=\dfrac{2(\quad)}{\sqrt{(\quad)+1}}$
$k(x)=2x-3$	$k(\quad)=2(\quad)-3$
$h(x)=\sqrt{x^2+5x}$	$h(\qquad)=\sqrt{(\qquad)^2+5(\qquad)}$ $=\sqrt{(\qquad\quad)^2+5(\qquad\quad)}$

Notation	$(f\circ g)(x)=f(g(x))$

Example:

1.) For the functions $f(x)=\sqrt{x}$ and $g(x)=x+2$ find

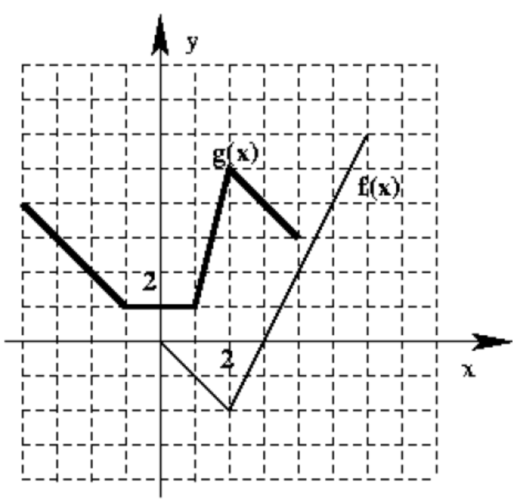
a.) $(f\circ g)(x)=$

b.) $(g\circ f)(x)=$

c.) $(f\circ g)(2)=$

d.) $(g\circ g)(x)=$

Example: For the functions f(x) and g(x) given in the graph find



a.) $(f\circ g)(2)=$

b.) $(f\circ g)(3)=$

c.) $(g\circ f)(2)=$

d.) $(g\circ f)(3)=$

e.) $(f\circ f)(4)=$

f.) $(g\circ g)(4)=$

Sec 1.3 The Limit of a Function

A. Limits

DEFN:	$\lim_{x \to a} f(x) = L$	The limit of f(x) as x approaches a, equals L.
	(Where is the functions value headed as x is "on its way" to a?)	
	$\lim_{x \to a^-} f(x)$	The limit of f(x) as x approaches a from the LEFT
	$\lim_{x \to a^+} f(x)$	The limit of f(x) as x approaches a from the RIGHT

B. Techniques of Solving Limits

1. Evaluation - When possible (without violating domain rules) "plug it in".

Example:

1.) $\lim_{x \to 3} x^2 =$

2.) $\lim_{x \to 1} \frac{1}{x} =$

2. Factoring/Manipulation (then Evaluation) - Factor expressions and cancel any common terms.

Example:

1.) $\lim_{x \to 4} \frac{x-4}{x^2-16} =$

2.) $\lim_{x \to 3} \frac{x^2-3x}{x^2-2x-3} =$

3. Table - Set up a table as x approaches the limit from the left and from the right.

Example:

1.) $\lim_{x \to 0} \frac{\sin x}{x} =$

4. Graphing - Graph the function and inspect. (Warning: Your graphing calculator might not always indicate a hole or small discontinuity in a graph. Be sure to always check the domain for restrictions.)

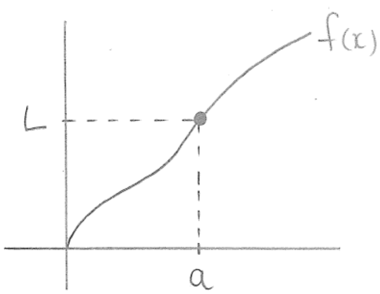
Example:

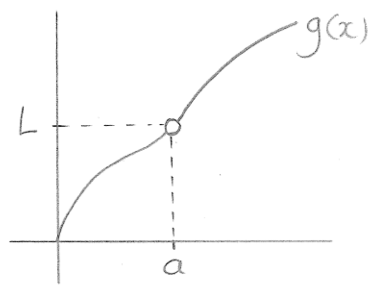
1.) $\lim_{x \to 0} \frac{\sin x}{x} =$

More Examples:

1.) $\lim_{x \to 0^+} \frac{1}{x} =$

2.)





$f(a) =$

$\lim_{x \to a} f(x) =$

$g(a) =$

$\lim_{x \to a} g(x) =$

3.)

$$f(x) = \begin{cases} 7 - x & \text{if } x \le -4 \\ x & \text{if } -4 < x \le 2 \\ (x-1)^2 & \text{if } x > 2 \end{cases}$$

$\lim_{x \to -4^-} f(x)$

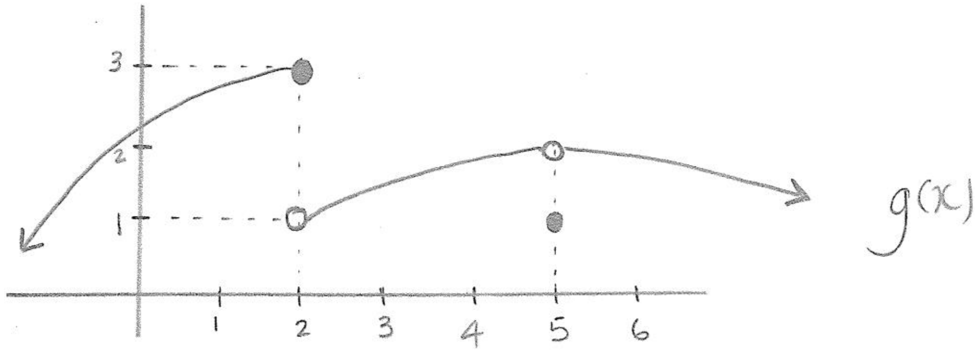
$\lim_{x \to -4^+} f(x)$

$\lim_{x \to -4} f(x)$

$\lim_{x \to 2^-} f(x)$

$\lim_{x \to 2^+} f(x)$

4.)



$g(2)=$

$g(5)=$

$\lim_{x\to 2^-} g(x)=$

$\lim_{x\to 5^-} g(x)=$

$\lim_{x\to 2^+} g(x)=$

$\lim_{x\to 5^+} g(x)=$

$\lim_{x\to 2} g(x)=$

$\lim_{x\to 5} g(x)=$

C. Average Velocity

DEFN: $$Velocity = \frac{Distance}{Time}$$

Example:

A ball is thrown up straight into the air with an initial velocity of 55 ft/sec, its height in feet t seconds is given by $y = 75t - 16t^2$.

a.) Find the average velocity for the period beginning when t=2 and lasting

(i) 0.1 seconds (i.e. the time period [2,2.1])

(ii) 0.01 seconds

(iii) 0.001 seconds

b.) Estimate the instantaneous velocity of the ball when t=2.

Sec 1.4 Calculating Limits

A. Limit Laws

Assume that f and g are functions and c is a constant.

1.) $\lim_{x\to a}\left(f(x)+g(x)\right)=\lim_{x\to a} f(x)+\lim_{x\to a} g(x)$

2.) $\lim_{x\to a}\left(f(x)-g(x)\right)=\lim_{x\to a} f(x)-\lim_{x\to a} g(x)$

3.) $\lim_{x\to a}\left(c\cdot f(x)\right)=c\cdot\left(\lim_{x\to a} f(x)\right)$

4.) $\lim_{x\to a}\left(f(x)\cdot g(x)\right)=\lim_{x\to a} f(x)\cdot \lim_{x\to a} g(x)$

5.) $\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right)=\frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$; $\lim_{x\to a} g(x)\neq 0$

6.) $\lim_{x\to a}\left(f(x)\right)^n=\left(\lim_{x\to a} f(x)\right)^n$

7.) $\lim_{x\to a} c=c$

8.) $\lim_{x\to a} x=a$

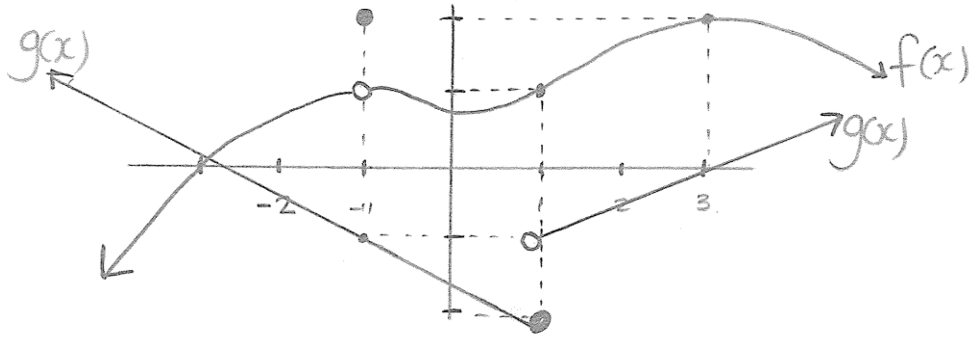
9.) $\lim_{x\to a} x^n=a^n$

10.) $\lim_{x\to a} \sqrt[n]{x}=\sqrt[n]{a}$

11.) $\lim_{x\to a} \sqrt[n]{f(x)}=\sqrt[n]{\lim_{x\to a} f(x)}$

Example:

1.)



a.) $\lim_{x\to 1^+} 3g(x) =$

b.) $\lim_{x\to 1} f(x) =$

c.) $\lim_{x\to 1} \left(f(x) + g(x)\right) =$

d.) $\lim_{x\to 3} \left(f(x) \cdot g(x)\right) =$

2.) Given $\lim_{x\to a} h(x) = 1$, $\lim_{x\to a} f(x) = 10$, $\lim_{x\to a} g(x) = 0$.

a.) $\lim_{x\to a} \dfrac{h(x)}{f(x)}$

b.) $\lim_{x\to a} f(x)^{-1}$

c.) $\lim_{x\to a} \sqrt{f(x)}$

d.) $\lim_{x\to a} \dfrac{1}{f(x) - g(x)}$

e.) $\lim_{x\to a} \dfrac{g(x)}{h(x)}$

B. Calculating Limits

Direct Substitution Property: If f is a polynomial or rational function and $a \in$ Domain, then $\lim_{x \to a} f(x) = f(a)$

Examples:

1.) $\lim_{x \to 2} x^2 + 3x + 1 =$

2.) $\lim_{x \to 5} \frac{2x+2}{x-3} =$

3.) $\lim_{\theta \to \pi} \cos\theta =$

4.) $\lim_{x \to 4} \frac{x^2 - 16}{x-4} =$

5.) $\lim_{x \to 0} \frac{(x+2)^2 - 4}{2x} =$

6.) $\lim_{x \to 0} \frac{\sqrt{x+16} - 4}{x} =$

Theorem We say that a limit exists when the limit from the left equals the limit from the right.

$$\lim_{x \to a} f(x) = L \quad \Leftrightarrow \quad \lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$$

Examples:

1.) $h(x) = \begin{cases} x^2 & if \;\; x \neq 2 \\ -1 & if \;\; x = 2 \end{cases}$ Find $\lim_{x \to 2} h(x)$

Theorem Squeeze Theorem: If $f(x) \le g(x) \le h(x)$ and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$ when x is near a, then $\lim_{x \to a} g(x) = L$

Examples:

1.) Find $\lim_{x \to 0} x \cdot \cos\left(\frac{1}{x}\right)$

More Examples:

1.) $\lim_{x \to -8^-} \frac{5x+40}{|x+8|} =$

$\lim_{x \to -8^+} \frac{5x+40}{|x+8|} =$

$\lim_{x \to -8} \frac{5x+40}{|x+8|} =$

2.) Evaluate $\lim_{x \to 0} \frac{\tan(5x)}{\sin(6x)} =$

Sec 1.5 Continuity

A. Definition of Continuity

DEFN: A function f is continuous at a number a if:

(i) $f(a)$ exists

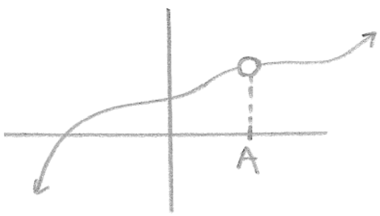
(ii) $\lim_{x \to a} f(x)$ exists

(iii) $\lim_{x \to a} f(x) = f(a)$

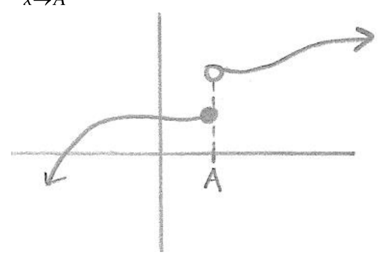
A function is defined as continuous only if it is continuous at every point in the domain of the function.

Examples: For each, determine whether the function is continuous (i.e. Is $\lim_{x \to A} f(x) = f(A)$?)

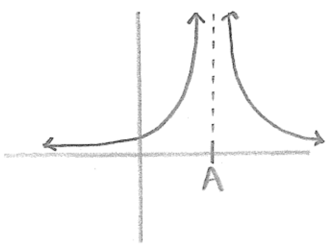
1.)



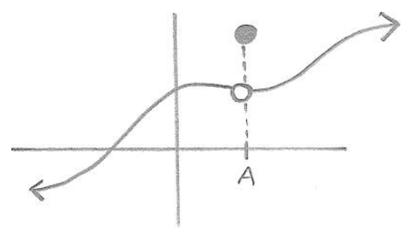
2.)



3.)



4.)



Examples: For each, determine whether the function is continuous. If not, where is the discontinuity?

1.) $f(x) = \dfrac{x^2 - x - 20}{x - 5}$

2.) $f(x) = \begin{cases} -6 - x & \text{if } x \le -3 \\ x & \text{if } -3 < x \le 3 \\ (x-1)^2 & \text{if } x > 3 \end{cases}$

3.) $h(x)=\begin{cases} x^2+1 & if\ x\neq 2 \\ 5 & if\ x=2 \end{cases}$

4.) $k(x)=\|x\|$ (Step function: i.e. $k(x)=\text{int}(x)$)

5.) For what value of the constant c is the function f continuous on $(-\infty,\infty)$ where

$$f(x)=\begin{cases} cx+7 & \text{if } x\in(-\infty,8] \\ cx^2-7 & \text{if } x\in(8,\infty) \end{cases}$$

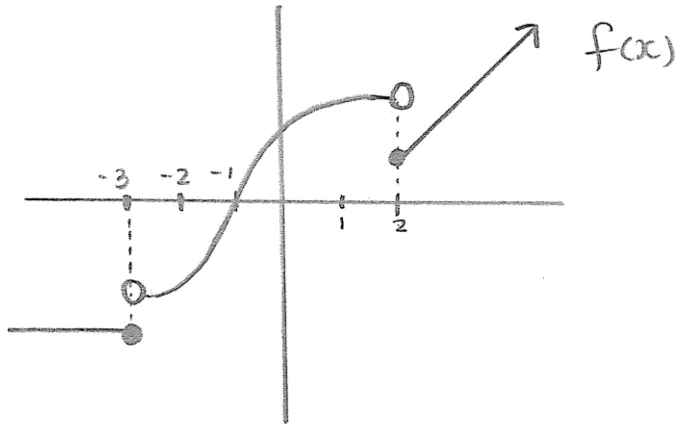
6.) Evaluate $\lim_{x\to 16}\dfrac{16-x}{4-\sqrt{x}}=$

DEFN: A function f is continuous from the RIGHT at a number a if: $\lim_{x \to a^+} f(x) = f(a)$

A function f is continuous from the LEFT at a number a if: $\lim_{x \to a^-} f(x) = f(a)$

Example:

1.)



Is f is continuous from the LEFT or RIGHT at

a.) $x = -3$

b.) $x = 2$

2.) Show that $f(x)$ has a jump discontinuity at $x = 9$ by calculating the limits from the left and right at $x = 9$.

$$f(x) = \begin{cases} x^2 + 5x + 5, & \text{if } x < 9 \\ 14, & \text{if } x = 9 \\ -4x + 4, & \text{if } x > 9 \end{cases}$$

<u>Theorem</u> If f and g are functions that are continuous at a number a, and c is a constant, then the following are also continuous at a:

(i) $(f + g)$

(ii) $(f - g)$

(iii) $(f \cdot g)$

(iv) $\left(\frac{f}{g}\right)$ if $g(a) \neq 0$

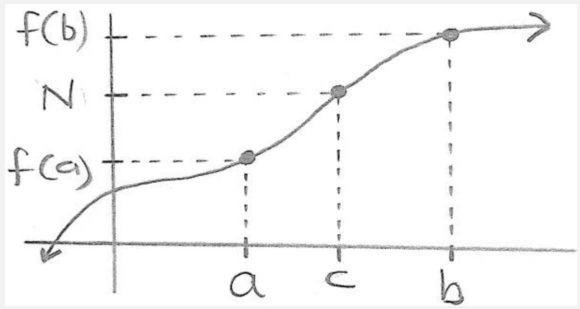
(v) $c \cdot f$ or $c \cdot g$

<u>Theorem</u> A **polynomial** function is continuous everywhere

A **rational** function is continuous everywhere it is defined

Theorem   Intermediate Value Theorem

If f is a function that is continuous on a closed interval $[a,b]$ where $f(a) \neq f(b)$ and N is a number such that $f(a) < N < f(b)$. Then there exist a number c such that $a < c < b$ and $f(c) = N$.



Examples:

1.) Show that $f(x) = x^2 - x - 2$ has a root on the interval $[1,3]$

2.) Let f be a continuous function such that $f(1) < 0 < f(9)$. Then the Intermediate Value Theorem implies that $f(x) = 0$ on the interval (A, B). Give the values of A and B.

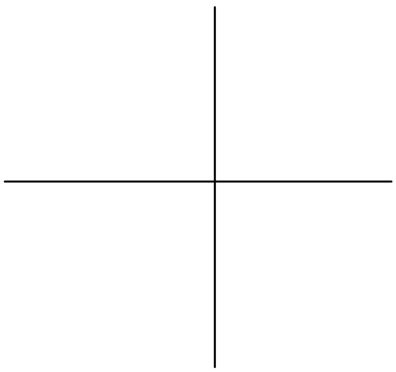
Sec 1.6 Limits Involving Infinity

A. Infinity vs. DNE

Recall from section 1.3 that $\lim_{x \to 0} \frac{1}{x^2}$ DNE since the function value kept increasing. Now we will be more descriptive; any value that keeps increasing is said to approach infinity (∞), and any value that keeps decreasing is said to approach negative infinity (-∞).

Examples:

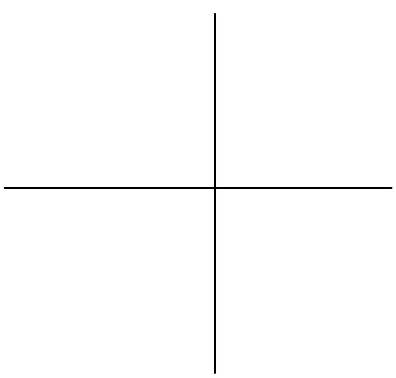
1.) Evaluate $\lim_{x \to 0} \frac{1}{x^2}$ using the graph and table method.



$\lim_{x \to 0^-} \frac{1}{x^2}$	
x	y
-0.1	
-0.01	
-0.001	

$\lim_{x \to 0^+} \frac{1}{x^2}$	
x	y
0.1	
0.01	
0.001	

2.) Evaluate $\lim_{x \to 0} \frac{1}{x}$ using the graph and table method.



$\lim_{x \to 0^-} \frac{1}{x}$	
x	y
-0.1	
-0.01	
-0.001	

$\lim_{x \to 0^+} \frac{1}{x}$	
x	y
0.1	
0.01	
0.001	

B. A Quick Review of Asymptotes

An asymptote is an imaginary line that the graph of a function approaches as the function approaches a restricted number in the domain or as it approaches infinity.

Locating Vertical Asymptotes

If $f(x)=\dfrac{p(x)}{q(x)}$ is a rational function, $p(x)$ and $q(x)$ have no common factors and n is a zero of $q(x)$, then the line $x=n$ is a vertical asymptote of the graph of $f(x)$.

Locating Horizontal Asymptotes.

Let $f(x)=\dfrac{p(x)}{q(x)}=\dfrac{a_nx^n+a_{n-1}x^{n-1}+....+a_1x+a_0}{b_mx^m+b_{m-1}x^{m-1}+....+b_1x+b_0}$

i. If n < m, then $y=0$ is the horizontal asymptote

ii. If n = m, then the line $y=\dfrac{a_n}{b_m}$ is the horizontal asymptote

iii. If n > m, there is NO horizontal asymptote. (But there will be a slant/oblique asymptote.)

Examples: For the following rational functions, find the vertical and horizontal asymptotes if any:

1.) $f(x)=\dfrac{16x^2}{4x^2+1}$

2.) $g(x)=\dfrac{x+8}{x^2-64}$

3.) $h(x)=\dfrac{x^3+7}{5x-2}$

4.) $k(x)=\dfrac{x^2-2x}{2-3x+x^2}$

C. Vertical Asymptotes

Vertical asymptotes occur when $\lim_{x \to a^-} f(x) = \pm\infty$ or $\lim_{x \to a^+} f(x) = \pm\infty$

The asymptote will be the line $x = a$.

Example: Evaluate the limit, find the asymptote and graph the function

1.) $\lim_{x \to 2} \frac{x+1}{3x-6}$

2.) $\lim_{x \to -4^+} \frac{x+6}{x+4}$

3.) $\lim_{x \to -4^-} \frac{x+6}{x+4}$

D. Limits as Infinity

A limit as the domain approaches infinity: $\lim_{x \to \infty} f(x)$

Finding Limits as Infinity of Rational Functions

i. Determine the degree of the denominator. (Let's say degree = P)

ii. Multiply both the numerator and denominator by $\frac{1}{x^P}$.

iii. Distribute/clean up algebra and continue evaluating the limit.

Example: Evaluate the limit.

1.) $\lim_{x \to \infty} \frac{6x^2 + 2x + 7}{8x + 2x^2}$

2.) $\lim_{x \to \infty} \frac{x^3 + 4x - 2}{6 - 2x^2}$

3.) $\lim_{x \to \infty} \frac{2x}{2x^2 + x - 1}$

<u>Conclusion</u>: For positive integers M *and* N *such that* $M > N$

1. Degree of the Numerator = Degree of the Denominator

$$\lim_{x \to \infty} \frac{Polynomail\ of\ Degree\ M}{Polynomail\ of\ Degree\ M} = Ratio\ of\ Leading\ Coeficient\,s$$

2. Degree of the Numerator > Degree of the Denominator

$$\lim_{x \to \infty} \frac{Polynomail\ of\ Degree\ M}{Polynomail\ of\ Degree\ N} = \pm\infty$$

3. Degree of the Numerator < Degree of the Denominator

$$\lim_{x \to \infty} \frac{Polynomail\ of\ Degree\ N}{Polynomail\ of\ Degree\ M} = 0$$

More Example: Evaluate the limit.

1.) $$\lim_{x\to\infty} \frac{3x-10}{\sqrt{16x^2+5}}$$

2.) $$\lim_{x\to-\infty} \frac{\sqrt{9x^2+x+1}}{2x+1}$$

3.) Find the horizontal asymptotes for the curve $y = \dfrac{12x}{\left(x^4+1\right)^{\frac{1}{4}}}$

4.) Find the vertical asymptotes for the curve $y = \dfrac{4x^3}{x+2}$

5.) $\lim_{x \to \frac{\pi}{2}^-} \tan x$

6.) $\lim_{x \to \frac{\pi}{2}^+} \tan x$

7.) $\lim_{x \to \infty} \sqrt{x^2 + 7x + 1} - x$