Sec 2.1 Derivatives and Rates of Change

A. Slope of Secant Functions

Recall: Slope = m = $\frac{rise}{run} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$. From this we are able to derive:

Slope of the Secant Line to a Function: $m = \frac{y_2 - y_1}{x_2 - x_1}$ or $m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$

Examples:

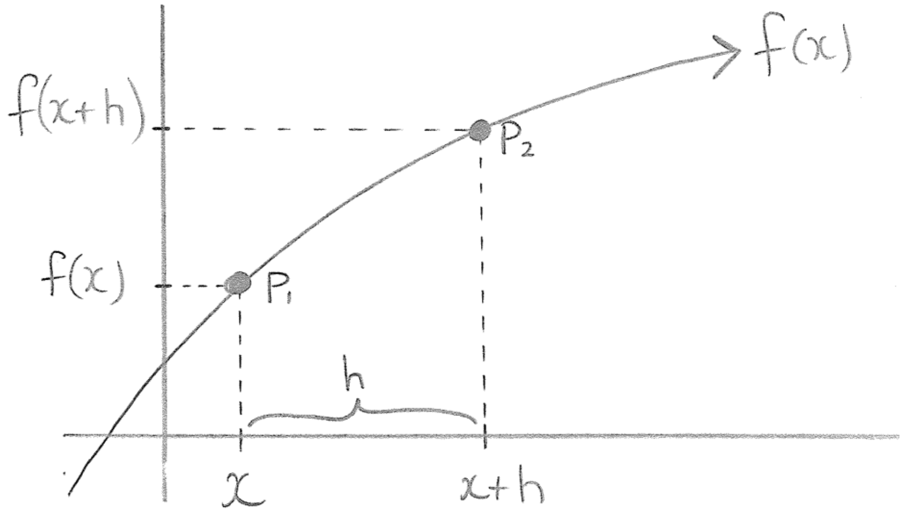
1. a.) Find the **slope** of the secant line to the function $f(x) = \sqrt{x}$ between $x = 1$ and $x = 9$

b.) Find the **equation** of the secant line to the function $f(x) = \sqrt{x}$ between $x = 1$ and $x = 9$

2.) Estimate the equation of the **tangent** line to the function $y = x^2$ at the point $(1, 1)$ by calculating the equation of the **secant** line between $x = 1$ and $x = 1.1$, between $x = 1$ and $x = 1.01$ and between $x = 1$ and $x = 1.001$.

A generalization of the previous example gives the definition: Slope = $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$

The slope of the tangent line can alternatively be calculated in the following way:



In order to get the two points P_1 and P_2 as close together as possible, we need for the space h→0.
So, the slope between P_1 and P_2 is:

$$m = \frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{(x+h) - (x)} = \frac{f(x+h) - f(x)}{h}$$

but as the space h→0, we have

$$m = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

B. Definition of Derivative

The definition of a derivative (a.k.a. the slope of the tangent function) is given as:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{or} \quad f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

*Note: In this section we will use the DEFINITION OF THE DERIVATE to calculate all derivatives. (This means we will be doing it the long way!)

Examples:

1.) Find the equation of the tangent line to the function $f(x) = x^2 - 3x + 1$ where $x = 5$.

2.) A person standing on top of a 200 ft tall building throws a ball into the air with a velocity of 96 ft/sec. The function $s(t) = -16t^2 + 96t + 200$ gives the ball's height above ground, t seconds after it was thrown. Find the instantaneous velocity of the ball at t = 2 seconds

3.) The position of a particle is given by the values of the table.

t(seconds)	0	1	2	3	4	5
s(feet)	0	14	47	51	86	103

Find the average velocity for the time period beginning when t=2 and lasting

1. 3 s (i.e. for the time interval [2,5])
2. 2 s
3. 1 s

4.) (a) The equation of the tangent line to the graph of $y = g(x)$ at $x = 3$ if $g(3) = -3$ and $g'(3) = 7$ is $y =$

(b) If the tangent line to $y = f(x)$ at (2, 10) passes through the point (0,4), then $f(2) =$
and $f'(2) =$

5.) $\lim_{h \to 0} \frac{\sqrt{81+h}-9}{h}$ represents the derivative of the function $f(x)=\sqrt{x}$ at the number $a =$

6.) $\lim_{x \to 6} \frac{2^x - 64}{x-6}$ represents the derivative of the function $f(x) =$_____ at the number $a =$

Sec 2.2 The Derivative of a Function

A. Definition of the Derivative

For a function $f(x)$ the derivative is $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

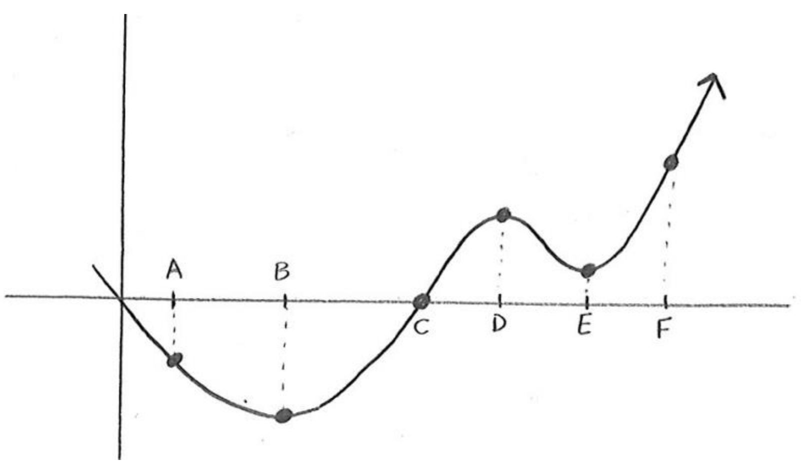
Examples: Using the definition of the derivative, find the derivative of the following functions:

1.) $f(x) = x^2 + 3x - 2$

2.) $g(x) = \sqrt{x} + 4$

3.) $k(x)=\dfrac{1}{2x+1}$

4.) Consider the graph for the function $f(x)$



Estimate the following:

a.) $f'(A)=$

b.) $f'(B)=$

c.) $f'(C)=$

d.) $f'(D)=$

e.) $f'(E)=$

f.) $f'(F)=$

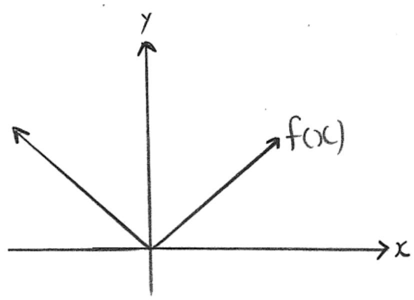
B. Differentiability

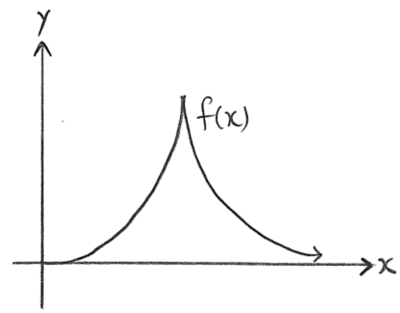
DEFN: We say that
- $f(x)$ is differentiable at a if $f'(a)$ exists
- $f(x)$ is differentiable on $(a\,,b)$ if it is differentiable on every point in $(a\,,b)$

Theorem: If f is differentiable at a , then f is continuous at a

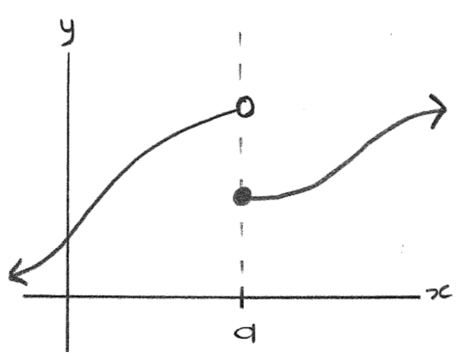
Non differentiable functions:

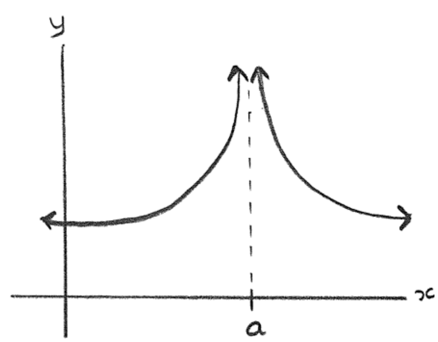
i. Any function with a "corner" or cusp



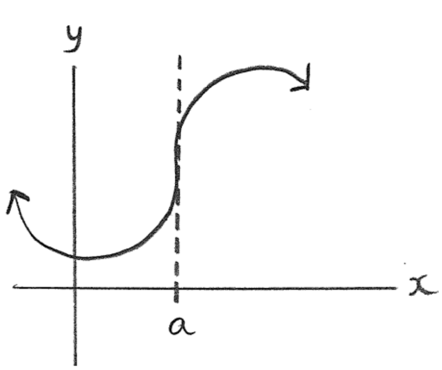


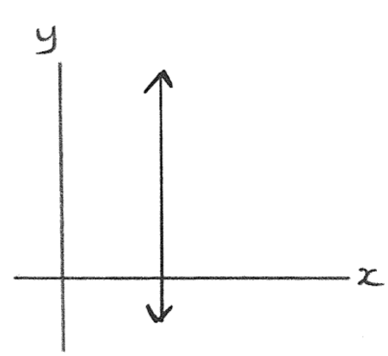
ii. Any function with a discontinuity





iii. Any function with a vertical tangent





C. Notation

Function	Derivative
$y =$	$y' =$ $\frac{dy}{dx} =$ (Leibniz notation)
$f(x)$	$f'(x)$
$F(x)$	$f(x)$

D. Higher Derivatives

(i.e. Finding the "derivative of a derivative" or taking the derivative multiple times.)

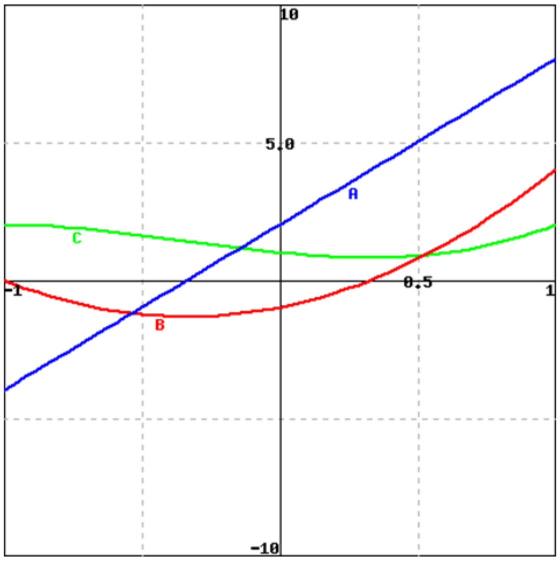
$$\frac{d\left(\frac{dy}{dx}\right)}{dx} = \frac{d^2y}{dx^2} = y'' = f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h}$$

Example:

1.) For the function $f(x) = x^2 + 3x - 2$, find the second derivative. (Recall from previous that $f'(x) = 2x + 3$)

More Examples:

1.)



Identify the graphs A (blue), B(red) and C (green) as the graphs of a function and its derivatives.

a.) The graph of the function is:

b.) The graph of the function's first derivative is:

c.) The graph of the function's second derivative is:

2.) Evaluate the limit $\lim_{x \to 1} \frac{x^3 - 1}{x^2 - 1}$

3.) Evaluate the limit $\lim_{x \to 0} \frac{\sqrt{x^2 + 36} - 6}{x^2}$

A. Properties and Formulas (The short way – Yeah!)

1. Basic Functions

Function	Derivative
$f(x)=c$ (Constant)	$f'(x)=0$
$f(x)=x$	$f'(x)=1$
$f(x)=cx$	$f'(x)=c$
$f(x)=x^n$	$f'(x)=nx^{n-1}$
$f(x)=cx^n$	$f'(x)=cnx^{n-1}$
$f(x)=c\cdot g(x)$	$f'(x)=c\cdot g'(x)$
$f(x)=g(x)+h(x)$	$f'(x)=g'(x)+h'(x)$
$f(x)=g(x)-h(x)$	$f'(x)=g'(x)-h'(x)$

Note: For the function $f(x)=g(x)\cdot h(x)$ we CAN NOT say that $f'(x)=g'(x)\cdot h'(x)$

For the function $f(x)=\dfrac{g(x)}{h(x)}$ we CAN NOT say that $f'(x)=\dfrac{g'(x)}{h'(x)}$

2. Trigonometric Functions

Function	Derivative
$f(x)=\sin(x)$	$f'(x)=\cos(x)$
$f(x)=\cos(x)$	$f'(x)=-\sin(x)$
$f(x)=\tan(x)$	$f'(x)=\sec^2(x)$
$f(x)=\csc(x)$	$f'(x)=-\csc(x)\cot(x)$
$f(x)=\sec(x)$	$f'(x)=\sec(x)\tan(x)$
$f(x)=\cot(x)$	$f'(x)=-\csc^2(x)$

Examples: Find and LABEL the derivatives of each of the following functions.

1.) $f(x)=8$

2.) $g(x)=8x$

3.) $k(x)=x^2$

4.) $h(x)=5x^3$

5.) $v(r)=\frac{3}{4}\pi r^3$

6.) $q(y)=y$

7.) $m(x)=\sqrt{x}$

8.) $a(x)=4x^{\frac{3}{2}}$

9.) $v(t)=\frac{1}{t}$

10.) $d(x)=\frac{1}{x^3}$

11.) $p(x)=\frac{x^3+4x^2}{x}$

12.) $d(x)=(x+3)^2$

More Examples

13.) $f(x)=x^3+4x^2-2x+4$ Find $f''(1)$

14.) $g(t)=\dfrac{4t^2+t+5}{\sqrt{t}}$ Find $g''(t)$

15.) $h(x)=x^5-2x^4+3x^3-x-6$ Find the first 5 derivatives of the function.

B. Normal and Tangent Lines to a Function

Normal Line: A line that is perpendicular to the tangent line.

Examples:

1.) Find the tangent and the normal lines to the function $f(x) = 4\cos(x)$ at $x = \frac{\pi}{3}$

2.) Find the horizontal tangent lines (lines with slope = 0) to the function $f(x) = 2x^3 + 3x^2 - 120x + 23$

C. Applications to Position, Velocity and Acceleration

If the motion/position function of a particle is known, we can find the velocity and acceleration functions in the following way.

- If the **position** of a particle is given by $f(x)$, then the **velocity** of the particle is given by $f'(x)$
- If the **velocity** of a particle is given by $g(x)$, then the **acceleration** of the particle is given by $g'(x)$
  (We can also say that If the position of a particle is given by $f(x)$, then the acceleration of the particle is given by $f''(x)$, the second derivative of the motion function.)

Alternative notation:

- **Position** of a particle $s(t)$
- **Velocity** of a particle $v(t)$
- **Acceleration** of a particle $a(t)$

Then $v(t) = s'(t)$ And $a(t) = v'(t) = s''(t)$

Example:

1.) A particle's **position** is described by the function $s(t) = 3t^3 - 144t$. (t is measures in seconds and $s(t)$ in feet.)

a. Find the velocity function.

b. Find the acceleration function.

c. Find the acceleration after 9 seconds.

d. Find the acceleration when the velocity is 0.

2.) A particle's **position** is described by the function $f(t)=t^3-9t^2+15t+10$. (t is measures in seconds and $s(t)$ in feet.)

a. Find the velocity function.

b. What is the velocity after 3 seconds?

c. When is the particle at rest?

d. When is the particle moving in a positive direction?

e. When is the particle slowing down?

f. Find the total distance traveled during the first 8 seconds.

3.) The area of a disc with radius r is $A(r) = \pi r^2$. Find the rate of change of the area of the disc with respect to its radius when $r = 5$.

More Examples:

1.) If $f(x) = (\frac{3}{4}x)^9$, then $f'(x) =$

2.) If $f(x) = \sqrt{x}(3x+0)$, then $f'(x) =$

3 a.) If $f(x) = 12\pi^2$, then $f'(x) =$

.

b.) If $f(x) = 12x^2$, then $f'(x) =$

c.) If $f(x) = 12\pi x^2$, then $f'(x) =$

4 If a ball is thrown vertically upward from the roof of 64 foot building with a velocity of 32 ft/sec, its height after t seconds is $s(t) = 64 + 32t - 16t^2$.

a.) What is the maximum height the ball reaches? .

b.) What is the velocity of the ball when it hits the ground (height 0)?

5 Evaluate the following limits:

a.) $\lim_{x\to 2} \frac{x^5 - 32}{x - 2} =$

b.) $\lim_{x\to 1} \frac{x^{615} - 1}{x - 1} =$

Sec 2.4 Product and Quotient Rules

A. Product Rule

$$f(x)=g(x)\cdot h(x) \text{ then } f'(x)=g'(x)\cdot h(x)+g(x)\cdot h'(x)$$

Alternative Notation: $\dfrac{d(g\cdot h)}{dx}=\left(\dfrac{dg}{dx}\right)\cdot(h)+(g)\cdot\left(\dfrac{dg}{dx}\right)$

In Plain English: The derivative of the product of two functions (which we will call the "first" function and the "second" function) is equal to the **derivative of the first, times the second, plus the first, times derivative of the second**.

Examples

1.) $f(x)=(2x^3+8x)\cdot(5x^4+17)$

We see that $f(x)$ consists of the product of two smaller functions, in this case $\underline{(2x^3+8x)}$ "the first" and $\underline{(5x^4+17)}$ "the second". So, the derivative then is:

$$f'(x)=\underbrace{(6x^2+8)}_{\textbf{Derivative of the first}} \times \underbrace{(5x^4+17)}_{\textbf{The second}} + \underbrace{(2x^3+8x)}_{\textbf{The first}} \times \underbrace{(20x^3)}_{\textbf{Derivative of the second}}$$

Note: You should leave the answer in this form unless we are asked to "clean up"
Again, do not forget to label your derivative

2.) $g(x)=x\cdot\sin(x)$

We see that $g(x)$ consists of the product of two smaller functions, in this case $\underline{x}$ "the first" and $\underline{\sin(x)}$ "the second".
So, the derivative then is: $g'(x)=(1)\cdot\sin(x)+x\cdot\cos(x)=\sin(x)+x\cdot\cos(x)$

More Examples: Find and LABEL the derivatives of each of the following functions.

1.) $f(x)=(x^4+\sqrt{x})\cdot(5x-1)$

2.) $f(x)=x^2\cos(x)$

3.) $f(x)=\sin(x)\cos(x)$

B. Quotient Rule

$$f(x)=\frac{g(x)}{h(x)} \text{ then } f'(x)=\frac{g'(x)\cdot h(x)-g(x)\cdot h'(x)}{[h(x)]^2}$$

Book Notation: $$\frac{d\left(\frac{g}{h}\right)}{dx}=\frac{\left(\frac{dg}{dx}\right)\cdot(h)-(g)\cdot\left(\frac{dg}{dx}\right)}{h^2}$$

In Plain English: The derivative of the quotient of two functions (which we will call the "top" function and the "bottom" function) is equal to the **derivative of the top, times the bottom, minus the top, times derivative of the bottom, all over the bottom squared.**

Examples

1.) $f(x)=\frac{(2x^3+8x)}{(5x^4+17)}$

We see that $f(x)$ consists of the quotient of two smaller functions, in this case $\underline{(2x^3+8x)}$ "the top" and $\underline{(5x^4+17)}$ "the bottom". So, the derivative then is:

$$f'(x)=\frac{\overbrace{(6x^2+8)}^{\text{Derivative of the top}} \times \overbrace{(5x^4+17)}^{\text{The bottom}} - \overbrace{(2x^3+8x)}^{\text{The top}} \times \overbrace{(20x^3)}^{\text{Derivative of the bottom}}}{\underbrace{(5x^4+17)^2}_{\text{The bottom squared}}}$$

Note: You should leave the answer in this form unless we are asked to "clean up"
Again, do not forget to label your derivative

2.) $g(x)=\frac{x}{\sin(x)}$

We see that $g(x)$ consists of the product of two smaller functions, in this case $\underline{x}$ "the top" and $\underline{\sin(x)}$ "the bottom".

So, the derivative then is: $g'(x)=\frac{(1)\cdot\sin(x)-x\cdot\cos(x)}{[\sin(x)]^2} = \frac{\sin(x)-x\cdot\cos(x)}{\sin^2(x)}$

More Examples: Find and LABEL the derivatives of each of the following functions.

1.) $f(x)=\frac{(x^4-8x)}{(2x-1)}$

2.) $g(x)=\frac{\sin x}{\cos x}$

3.) $l(x)=\frac{\sin x}{\sqrt{x}}\cdot(x^2+2x)$

4.) Suppose $f(\pi/6)=7$ and $f'(\pi/6)=-5$, and let $g(x)=f(x)\cos x$ and $h(x)=\frac{\sin x}{f(x)}$

$g'(\pi/6)=$

$h'(\pi/6)=$

5.)

x	1	0	7	-2	-1
$f(x)$	-5	-1	-407	7	1
$g(x)$	-3	-2	-9	0	-1
$f'(x)$	-7	-2	-163	-10	-3
$g'(x)$	-1	-1	-1	-1	-1

a. $(fg)'(-1)$

b. $f(-1)/(g(-1)+5)$

c. $(f+g)'(-1)$

d. $(f-g)'(-1)$

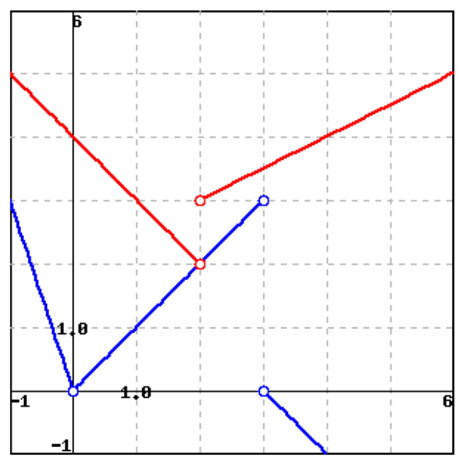
e. $(fg)'(7)$

f. $\left(\dfrac{g}{f}\right)'(0)$

6.) The graphs of the function f (given in blue) and g (given in red) are plotted above. Suppose that $u(x) = f(x)g(x)$ and $v(x) = f(x)/g(x)$. Find each of the following:

$u'(1) =$

$v'(1) =$



f is the bottom function

g is the top function

7.) Given that

$$f(x) = x^{12}h(x)$$
$$h(-1) = 3$$
$$h'(-1) = 6$$

Calculate $f'(-1)$

A. The Chain Rule

$$[h(g(x))]' = h'(g(x)) \cdot g'(x)$$

Alternative Notation:

$f(x) = (h \circ g)(x) = h(g(x))$ then $f'(x) = [h'(g(x))] \cdot [g'(x)]$

$$[h(g(x))]' = \frac{d(h(g))}{dx} = \left(\frac{dh}{dg}\right) \cdot \left(\frac{dg}{dx}\right)$$

In Plain English: First, identify which function is on the "outside" and which is on the "inside". (For the composition $(f \circ g)(x) = f(g(x))$ we say that f is on the "outside" and g is on the "inside".) The derivative of this composition is equal to **the derivative of the outside (leave the inside alone) times the derivative of the inside**.

Examples

1.) $f(x) = \sin(5x^5)$

First, let us identify which is the "outside" and which is on the "inside": Here $\underline{\sin(\ldots)}$ is the "outside" (i.e. "sin of something") and $\underline{5x^5}$ is the "inside".

Derivative of the "outside" is $\cos(\ldots)$, and if we the leave the inside alone, this will be $\cos(5x^5)$

Derivative of the "inside" is $25x^4$

$$f'(x) = \underbrace{\left(\cos(5x^5)\right)}_{\text{Derivative of the outside (leave the inside alone)}} \times \underbrace{\left(25x^4\right)}_{\text{Derivative of the inside}}$$

Note: This style of answer should only be "cleaned up" if you are given specific instructions to so (or if you have to compare it to a list of multiple choice answers)!
Again, do not forget to label your derivative

More Examples

2.) $g(x)=\sin^2(x)=[\sin(x)]^2$

We see that $g(x)$ consists of $\underline{[\ldots]^2}$ as the "outside" and $\underline{\sin(x)}$ as the "inside".

So, the derivative of the "outside" is $2[\ldots]$ and derivative of the "inside" is $\cos(x)$

$$\Rightarrow \quad g'(x)=2[\sin(x)]\cdot[\cos(x)]$$

Examples: Find and LABEL the derivatives of each of the following functions.

1.) $f(x)=\tan(6x^3)$

2.) $k(x)=\tan(\sin x)$

3.) $l(x)=\sqrt{x^2+1}$

4.) $f(x)=(x^2+2x)^8$

5.) $r(s)=\dfrac{1}{\sqrt[3]{s^4+s}}$

B. Combinations of Product, Quotient and Chain Rule

In many problems we need to use a combination of the Product, Quotient and Chain Rule to find a derivative. Here we will work through lots of examples.

Examples: Find and LABEL the derivatives of each of the following functions. Do not clean up unless otherwise indicated.

1.) $g(x) = \sin(x^2) \cdot (8x^3 - 1)$

2.) $f(x) = \sin^3(x^2) \cdot$

3.) $g(x) = (x - \cos^2(x))^4$

4.) $r(\theta) = \dfrac{\cos(2\theta - 4\pi)}{\theta^2}$

5.) $a(x) = x^2 \cdot \sqrt{2x^3 + 1}$

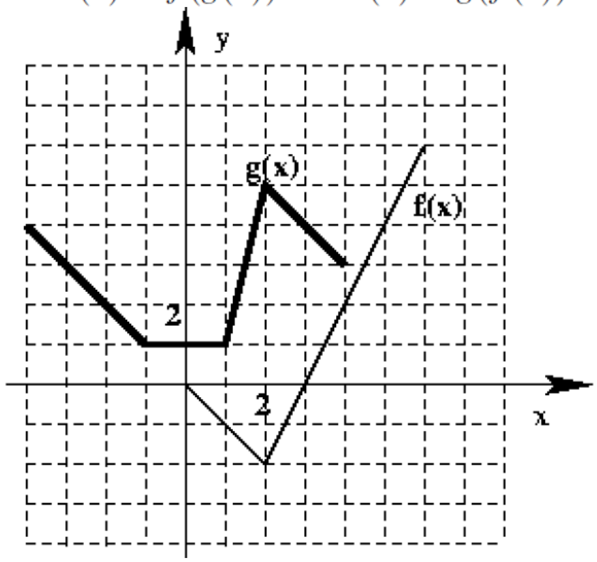
6.) $f(x) = \frac{5x^3}{(x^2 + 5x)^3}$

7.) If $f(x) = \cos(a^8 + x^8)$, then $f'(x) =$

8.) If $f(x) = \cos(x \sin x)$, find $f'(x) =$

9.) Let $F(x) = f(g(x))$, where $f(-7) = 2$, $f'(-7) = 2$, $f'(3) = 15$, $g(3) = -7$, and $g'(3) = -8$, find $F'(3)$

10.) If f and g are the functions whose graphs are shown below, let $u(x) = f(g(x))$ and $v(x) = g(f(x))$.



Find $u'(3)$

and $v'(3)$

11.) Let $F(x) = f(x^6)$ and $G(x) = (f(x))^6$. You also know that $a^5 = 11$, $f(a) = 2$, $f'(a) = 8$, $f'(a^6) = 15$.

Find $F'(a)$

and $G'(a)$

Sec 2.6 Differentiation of Implicit Functions

Recall from the previous week, that when we take the derivative of $y = f(x)$ then $y' = f'(x)$ where $f(x)$ is a function in terms of x (i.e. the only variable in the function is x)

Example: If $y = x^3 + 2x$ then $y' = \frac{dy}{dx} = 3x^2 + 2$

So, in other words, to take a derivative this way, we have to have the equation solved for "y".

Example: If $x = x^3 + 2x - y$. Here we first have to solve for y. So $y = x^3 + 2x - x \quad \Rightarrow \quad y = x^3 + x$

then $y' = \frac{dy}{dx} = 3x^2 + 1$

Example: If $yx = x^3 + 2xy^2 - y$. Again, we have to solve for y in order to take a derivative in the way that we have learnt in the preceding chapters. However, (as you can see in this case) it is not always easy/possible to do so.

HENCE: Implicit Differentiation!

A. Implicit Differentiation

Implicit Differentiation: Differentiation of a function where one variable (typically y) is not explicitly expressed a s a function of another variable (typically x).

<u>Here's how it works:</u>

- It is important to pay attention to the notation. If we are given an equation in terms of x and y, and asked to find y' or $\frac{dy}{dx}$, we need to see that we are finding the derivative of **y**, with respect to **x**.
- We will treat both x and y like a variable, and take derivatives of each, but;
- When we take a derivative of a term containing "x" we will proceed as usual
- When we take a derivative of a term containing "y" we will proceed as usual AND then also multiply the derivative of that term by $\frac{dy}{dx}$ (or y').
- We will use product, quotient and chain rules as needed.
- After differentiating, solve for (i.e. isolate) y' or $\frac{dy}{dx}$,

Example: Find $\frac{dy}{dx}$ for $x = x^3 + y^2$.

$$\frac{d(x)}{dx} = \frac{d(x^3)}{dx} + \frac{d(y^2)}{dx} \quad \Rightarrow \quad 1 = 3x^2 + 2y \cdot \frac{dy}{dx}$$

Since we are trying to find $\frac{dy}{dx}$, isolate $\frac{dy}{dx}$ in our equation: $1 - 3x^2 = 2y \cdot \frac{dy}{dx} \quad \Rightarrow \quad \frac{1-3x^2}{2y} = \frac{dy}{dx}$

Example: Find $\frac{dy}{dx}$ for $x = 4x^3 + y^2 - 8y$.

$$\frac{d(x)}{dx} = \frac{d(4x^3)}{dx} + \frac{d(y^2)}{dx} - \frac{d(8y)}{dx} \quad \Rightarrow \quad 1 = 12x^2 + 2y \cdot \frac{dy}{dx} - 8 \cdot \frac{dy}{dx}$$

Since we are trying to find $\frac{dy}{dx}$, isolate $\frac{dy}{dx}$ in our equation:

$$1 - 12x^2 = 2y \cdot \frac{dy}{dx} - 8 \cdot \frac{dy}{dx} \quad \Rightarrow \quad 1 - 12x^2 = \frac{dy}{dx}(2y - 8) \quad \Rightarrow \quad \frac{1 - 12x^2}{2y - 8} = \frac{dy}{dx}$$

Example: Find y' for $x = x^3 y^2 - 3y^3$. (Notice that in this problem we have $x^3 y^2$ - a product of x and y. Here we will have to use the product rule.)

$$\frac{d(x)}{dx} = \frac{d(x^3 y^2)}{dx} - \frac{d(3y^3)}{dx} \quad \Rightarrow \quad 1 = [(3x^2)(y^2) + (x^3)(2y \cdot y')] - 9y^2 \cdot y'$$

$$\Rightarrow \quad 1 - (3x^2)(y^2) = (x^3 \cdot 2y) \cdot y' - (9y^2) \cdot y' \quad \Rightarrow \quad 1 - (3x^2)(y^2) = y'(x^3 \cdot 2y - 9y^2)$$

$$\Rightarrow \quad \frac{1 - 3x^2 y^2}{2x^3 y - 9y^2} = y'$$

Examples: Find $\frac{dy}{dx}$ for the following:

1.) $x^2 + 2y^2 - 11 = 0$

2.) $y^2x - \frac{5y}{x+1} + 3x = 4$

3.) Find y' for $2y + 5 - x^2 - y^3 = 0$ and evaluate at $(2,-1)$

4.) Find $\frac{dA}{dt}$ for $A = \pi r^2$

5.) Find $\frac{dV}{dt}$ for $V = \frac{1}{3}\pi r^2 h$

6.) If $f(x) + x^7[f(x)]^3 = 11$ and $f(2) = 6$, find $f'(2) =$

7.) Use implicit differentiation to find an equation of the tangent line to the curve
line to the curve $4x^2 - 4xy - 1y^3 = 84$ at the point $(1, -4)$ of the form $y = mx + b$
$m =$ _____ and $b =$ _____

Sec 2.7 Related Rates

Before getting started with Related Rates, let us re-visit the following items first: Notation, Implicit Differentiation and Geometric Formulas

A. Notation

$$f'(x) \Leftrightarrow y' \Leftrightarrow \frac{dy}{dx}$$

Although all of the above notations are equivalent, we will use Leibniz's notation $\left(\frac{dy}{dx}\right)$ in this section, because it is more descriptive than the other forms. Leibniz's notation tells us specifically what we are taking a derivative of (in this case the function y) and what we are taking the derivative with respect to (w.r.t.) – i.e. what is the variable in the function (in this case x.)

B. Implicit Differentiation

Again, we will have to pay close attention to notation here. In equations with multiple variables, we will be asked to find derivatives of specific parts of the equations with respect to specific variables (that may or may not be part of the equation!).

For Example: Consider the equation for the circle: $r^2 = x^2 + y^2$

- We would like to find $\frac{dr}{dt}$. This means, we are trying to find the derivative of **r** with respect to **t**. I.e. take a derivative of each term with respect to **t**. (If a term is/contains a **t**, just take a derivative as usual. If a term contains a variable other than **t**, follow the usual rules for implicit differentiation.)

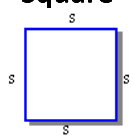
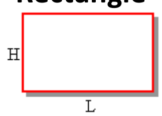
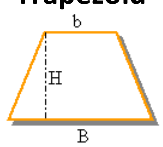
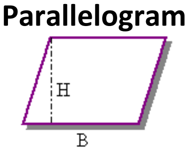
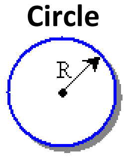
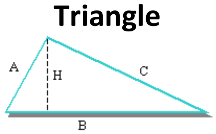
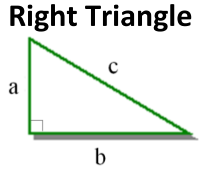
  So, $(2r)\frac{dr}{dt} = (2x)\frac{dx}{dt} + (2y)\frac{dy}{dt} \quad \Rightarrow \quad \frac{dr}{dt} = \frac{(2x)\frac{dx}{dt} + (2y)\frac{dy}{dt}}{2r}$

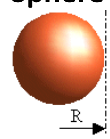
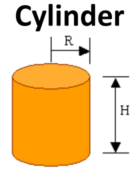
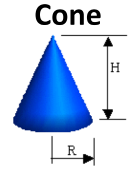
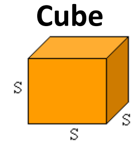
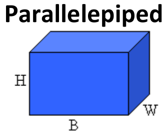
- We would like to find $\frac{dx}{dt}$.

  So, $(2r)\frac{dr}{dt} = (2x)\frac{dx}{dt} + (2y)\frac{dy}{dt} \quad \Rightarrow \quad \frac{dx}{dt} = \frac{(2r)\frac{dr}{dt} - (2y)\frac{dy}{dt}}{2x}$

C. Geometric Formulas

* You are responsible for knowing these formulas for all tests and the final exam*

2-Dimentional Shapes	
Shape	**Perimeter/Circumference and Area**
Square 	$P = 4S$ $A = S^2$
Rectangle 	$P = 2H + 2L$
Trapezoid 	$A = \frac{H(B+b)}{2}$
Parallelogram 	$A = BH$
Circle 	$C = 2\pi R$ $A = \pi R^2$
Triangle 	$A = \frac{BH}{2}$
Right Triangle 	$c^2 = a^2 + b^2$

3-Dimentional Shapes	
Shape	
Sphere 	$SA = 4\pi R^2$ $V = \frac{4}{3}\pi R^3$
Cylinder 	$SA = 2\pi RH + 2\pi R^2$ $V = \pi R^2 H$
Cone 	$V = \frac{\pi R^2 H}{3}$
Cube 	$SA = 6S^2$ $V = S^3$
Rectangular Parallelepiped 	$SA = 2HB + 2BW + 2HW$ $V = BHW$

D. Related Rate Problems

1. If $z^2 = x^2 + y^2$, $\frac{dx}{dt} = 9$, and $\frac{dy}{dt} = 5$, Find $\frac{dz}{dt}$ when $x = 2$ and $y = 5$

2. Suppose oil spills from a ruptured tanker and spreads in a circular pattern. If the radius of the oil spill increases at a constant rate of 1.5 m/s, how fast is the area of the spill increasing when the radius is 19 m?

Hint: The word **rate = derivative**. Pay attention to units to find out which rate is given asked for.
(Example, "rate of 1.5 m/s" – meters per second = unit of **length** per unit of **time =** $\frac{d(length)}{d(time)} = \frac{d(r)}{d(t)}$)

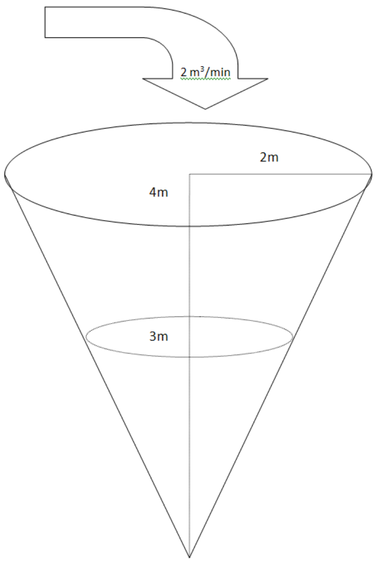
3. A fireman is on top of a 75 foot ladder that is leaning against a burning building. If someone has tided Sparky (the fire dog) to the bottom of the ladder and Sparky takes off after a cat at a rate of 6 ft/sec, then what is the rate of change of the fireman on top of the ladder when the ladder is 5 feet off the ground?

4. A street light is mounted at the top of a 11 ft tall pole. A woman 6 ft tall walks away from the pole with a speed of 8 ft/sec along a straight path. How fast is the tip of her shadow moving when she is 50 ft from the base of the pole?

5. If a snowball melts so that its surface area decreases at a rate of $.01\frac{cm^2}{\min}$, find the rate at which the diameter decreases when the diameter is 8cm.

6. At noon, ship A is 30 miles due west of ship B. Ship A is sailing west at 25 mph and ship B is sailing north at 18 mph. How fast (in mph) is the distance between the ships changing at 5 PM?

7. Water pours into in inverted cone at a rate of 2 m^3/min. It the cone has a radius of 2 m and a height of 4 m, find the rate at which the water level is rising when the water is 3 m deep.



8.) A boat is pulled into a dock by means of a rope attached to a pulley on the dock. The rope is attached to the front of the boat, which is 7 feet below the level of the pulley. If the rope is pulled through the pulley at a rate of 20 ft/min, at what rate will the boat be approaching the dock when 120 ft of rope is out?

* Examples on this topic available at Justmathtutoring.com > Free Calculus Videos > Related Rates - … *

A. Differentials

Slope = $\dfrac{\Delta y}{\Delta x}$

Slope of the tangent line = $\dfrac{dy}{dx}$

$\Longrightarrow \quad \dfrac{\Delta y}{\Delta x} \approx \dfrac{dy}{dx}$

Also, $\dfrac{dy}{dx} = f'(x) \quad \Longrightarrow \quad dy = [f'(x)] \cdot dx$

Example: Find the differential of y given that:

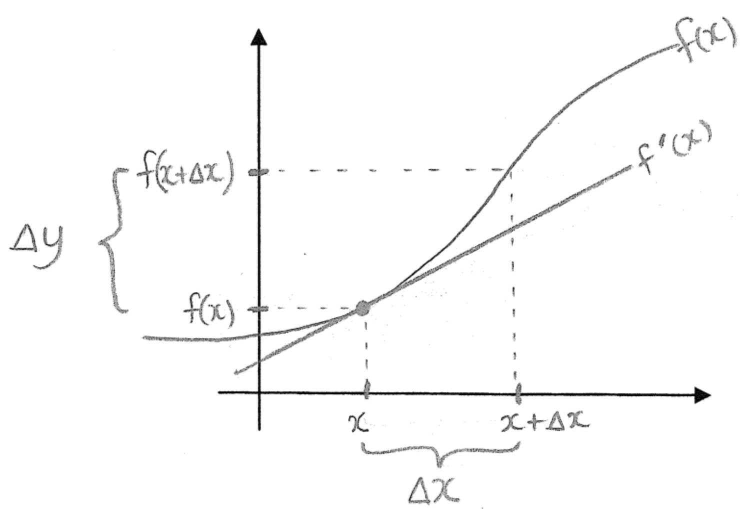
1.) $y = 5x^3$

2.) $y = \dfrac{3x^2 + 4x - 5}{5\sin(x) + 2x}$

Note:

$\Delta x \approx dx$ and $\Delta y \approx dy$

$$dx \approx \Delta x = (x + \Delta x) - (x)$$
$$dy \approx \Delta y = f(x + \Delta x) - f(x)$$



Example: $y = x^3$

1a.) Calculate Δy for $x = 2$ to $x = 2.01$

b.) Calculate dy for $x = 2$ to $x = 2.01$

2.) $V = \frac{4}{3}\pi r^3$ (volume of a sphere)

a.) Use differentials to approximate the change in volume when going from $r = 3\,ft$ to $r = 2.8\,ft$

(Find dV : Start by finding $\frac{dV}{dr}$)

b.) Find the actual change in volume when going from $r = 3\,ft$ to $r = 2.8\,ft$

(Find ΔV)

B. Linearization

Definition: The linearization of a function $f(x)$ at a fixed point a is given by the formula

$$L(x) = f(a) + f'(a)(x-a)$$

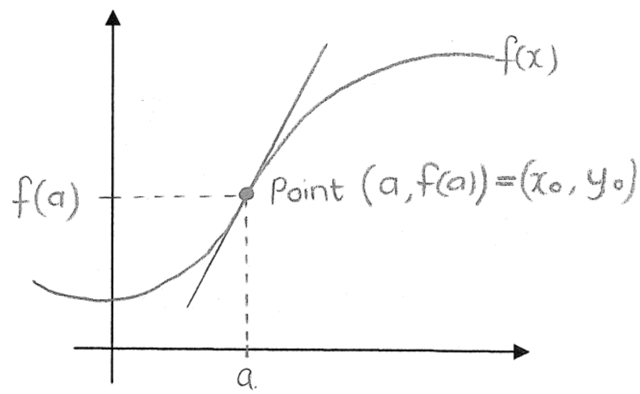
Slope of the tangent line = $m_T = f'(a)$

Point Slope: $(y - y_o) = m_T(x - x_0)$

$$(y - f(a)) = f'(a)(x-a)$$

$$y = f(a) + f'(a)(x-a)$$

$$L(x) = f(a) + f'(a)(x-a)$$



Examples:

1.) Find the linear approximation of $f(x) = \cos(5x)$ at $a = \dfrac{\pi}{2}$

2.) Use linearization techniques to approximate $\sqrt{16.1}$

More Examples:

3.) Find the linear approximation of $f(x) = \sqrt{4-x}$ at $a = 0$ and use it to approximate $\sqrt{3.9}$ and $\sqrt{4.1}$.

4.) Use a linear approximation to approximate 2.001^6 as follows:
The linearization $L(x)$ to $f(x) = x^6$ at $a = 2$ can be written in the form $L(x) = mx + b$
Using this, the approximation for 2.001^6 is

5.) The edge of a cube was found to be 60 cm with a possible error of 0.5 cm. Use differentials to estimate:

(a) the maximum possible error in the volume of the cube

(b) the relative error in the volume of the cube

(c) the percentage error in the volume of the cube

6.) A 13 foot ladder is leaning against a wall. If the top slips down the wall at a rate of 4 ft/s, how fast will the foot of the ladder be moving away from the wall when the top is 8 feet above the ground?