Sec 3.1 Exponential Functions

A. Limit Rules

1. $\lim_{r\to x} a^r = a^x$ 2. If $a>1$, then $\lim_{x\to\infty} a^x = \infty$ and $\lim_{x\to-\infty} a^x = 0$ 3. If $0<a<1$, then $\lim_{x\to\infty} a^x = 0$ and $\lim_{x\to-\infty} a^x = \infty$	4. $\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e$ 5. $\lim_{x\to\infty} e^x = \infty$ and $\lim_{x\to-\infty} e^x = 0$

Examples:

1. Starting with the graph of $f(x)=9^x$, write the equation of the graph that results from:

a.) Shifting $f(x)$ 9 units upward.

b.) Shifting $f(x)$ 7 units to the right.

c.) Reflecting $f(x)$ about the x-axis.

2. The domain of the function $f(x)=\dfrac{16}{1-e^x}$

3. Find the exponential function $f(x)=Ca^x$ whose graph goes through the points (0 , 5) and (2 , 20).

4. Evaluate the following limit $\lim_{x\to\infty} 0.77^x =$

5. Evaluate the following limit $\lim_{x\to\infty} e^{-x^2} =$

6. Evaluate the following limit $\lim_{x\to\infty} \frac{e^{5x}+e^{-5x}}{e^{5x}-e^{-5x}} =$

7. Evaluate the following limit $\lim_{x\to\infty} \left(e^{-2x}\cos x\right) =$

Sec 3.2 Logarithmic Functions

A. Limit Rules

1. $\lim_{x\to 0^+} \ln(x) = -\infty$ 2. $\lim_{x\to\infty} \ln(x) = \infty$	3. $\lim_{x\to 0^+} \log(x) = -\infty$ 4. $\lim_{x\to\infty} \log(x) = \infty$

$$\left(f^{-1}\right)'(a) = \frac{1}{f'\left(f^{-1}(a)\right)}$$

Examples:

1. $\lim_{x\to 3^+} \log\left(x^2 - 5x + 6\right) =$

2. $\lim_{x\to 0^+} \log(\sin x) =$

3. $\lim_{x\to\infty} \left[\log\left(1 + x^2\right) - \log(1 + x)\right] =$

4. If f is one-to-one and $f(3)=11$, then

a.) $f^{-1}(11)=$

b.) $[f(3)]^{-1}=$

5. Find the inverse for each of the following:

a.) $f(x)=\dfrac{4x-12}{19x+15}$

b.) $h(x)=e^{9x+3}$

c.) $f(x)=\ln(13x+10)$

6. For $f(x) = x^3 + 4x + 4$, find $(f^{-1})'(4) =$

7. Suppose f^{-1} is the inverse function of a differentiable function f and $f(3) = 4$, $f'(3) = \frac{7}{4}$ then $(f^{-1})'(4) =$

8. If $\ln(a) = 2$, $\ln(b) = 3$, and $\ln(c) = 5$, evaluate $\ln\left(\sqrt{b^{-4}c^{-4}a^{-3}}\right) =$

9. Solve each equation for x:

a.) $3^{x-4} = 9$

b.) $\ln(x) + \ln(x-1) = 4$

A. Derivatives

1.) $\frac{d}{dx}(\log_a x) = \frac{1}{x \cdot \ln a}$	3.) $\frac{d}{dx}(a^x) = a^x \cdot \ln a$
2.) $\frac{d}{dx}(\ln x) = \frac{1}{x}$	4.) $\frac{d}{dx}(e^x) = e^x$

Examples: Find the derivative of each

1.) $f(x) = \ln(x^3 + 10)$

2.) $y = e^{-5x} \cdot \cos(3x)$

3.) $f(x) = \ln\left(\sqrt{\frac{4x-6}{x+1}}\right)$

4.) $f(x) = 3x^6 \cdot e^{2x}$

5.) $f(x) = \log_3(5x^2 + 4)$

6.) $f(x) = 7^{5x}$

B. Logarithmic Differentiation

Examples: Find the derivative of each

1.) $y = x^{\sqrt{x}}$

2.) $y = \dfrac{(9x+4)^7 (x^3 - 5)^3}{\sqrt{3x-1}}$

Sec 3.4 Exponential Growth and Decay

A. Population Growth

$$P(t) = P(0) \cdot e^{kt}$$

Where:

P(t) = Population after t years
P(0) = Initial Population
K = Growth constant
T = Time

A bacteria culture initially contains 600 cells and grows at a rate proportional to its size. After 5 hours the population has increased to 620.

a.) Find an expression for the number of bacteria after t hours.

b.) Find the number of bacteria after 7 hours.

c.) Find the rate of growth after 7 hours. (Remember: Rate = Derivative)

d.) When will the population reach 4000?

B. Half Life

$$P(t) = P(0) \cdot e^{kt}$$

Where:

P(t) = Population after t years
P(0) = Initial Population
K = Growth constant
T = Time

The half-life of cesium-137 is 30 years. Suppose we have a 900-mg sample.

a.) Find the mass that remains after t years. (Find an expression for the mass that remains after t years.)

b.) How much of the sample remains after 150 years?

c.) After how long will only 4 mg remain?

C. Newton's Law of Cooling

	Alternatively
$$T(t) = (T_0 - T_s) \cdot e^{kt} + T_s$$ Where: T(t) = Temperature after time t; T_s = Temperature of surrounding area; T_0 = Initial temperature of object; K = Growth constant; T = Time	$$T(t) = C \cdot e^{kt} + T_s$$ Where: T(t) = Temperature after time t; T_s = Temperature of surrounding area; C = Initial temp - surrounding temp; K = Growth constant; T = Time

A roast turkey is taken from an oven when its temperature has reached 175 Fahrenheit and is placed on a table in a room where the temperature is 65 Fahrenheit.

a.) If the temperature of the turkey is 155 Fahrenheit after half an hour, what is its temperature after 45 minutes?

b.) When will the turkey have cooled to 110 Fahrenheit?

D. Interest

Compound Interest	Compound Interest
$$A = P\left(1+\frac{r}{n}\right)^{nt}$$ Where: A = Future Value P = Initial Value r = Interest rate n = Number of times per year compounded t = Time in years	$$A = P \cdot e^{rt}$$ Where: A = Future Value P = Initial Value r = Interest rate t = Time in years

If 8000 dollars is invested at 9% interest, find the value of the investment at the end of 5 years if interest is compounded

a.) annually

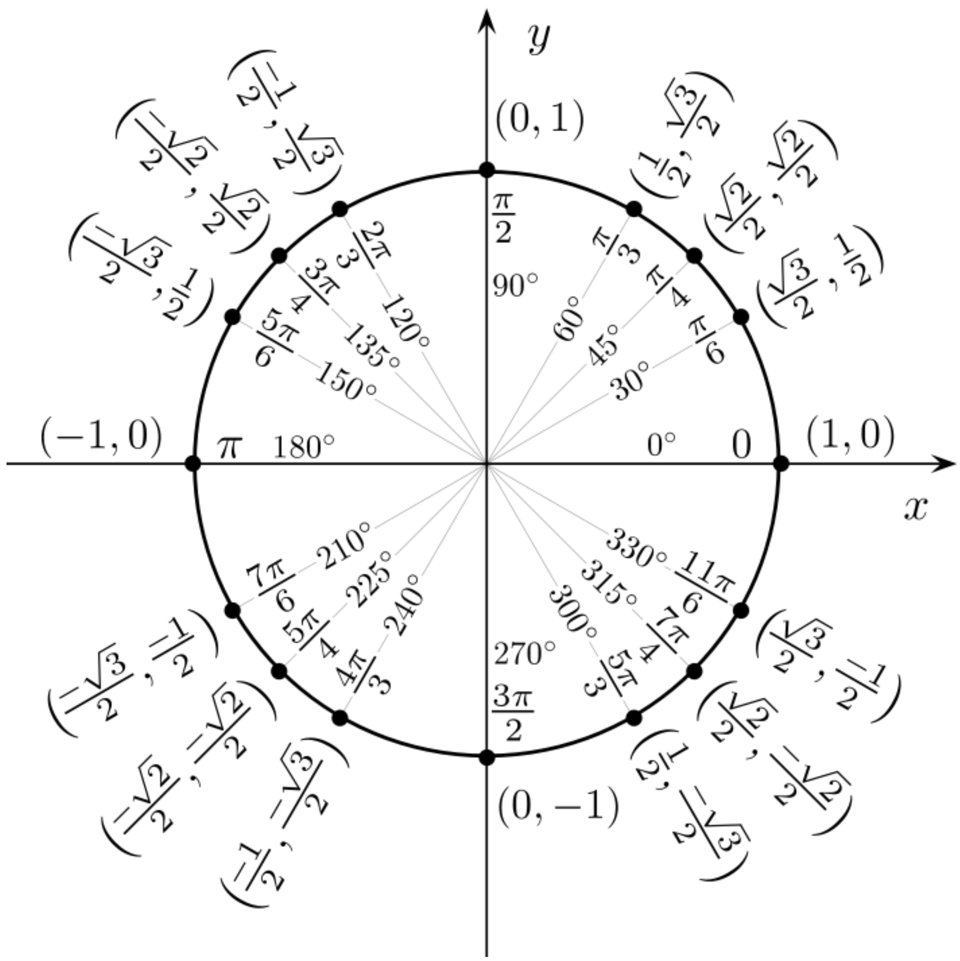
b.) quarterly

c.) monthly

d.) continuously

Sec 3.5 Inverse Trigonometric Functions

A. Unit Circle and Common Values



Degrees	Radians	sin θ	cos θ	tan θ	cot θ	sec θ	csc θ
0°	0	0	1	0	Undefined	1	Undefined
30°	π/6	1/2	$\sqrt{3}/2$	$\sqrt{3}/3$	$\sqrt{3}$	$2\sqrt{3}/3$	2
45°	π/4	$\sqrt{2}/2$	$\sqrt{2}/2$	1	1	$\sqrt{2}$	$\sqrt{2}$
60°	π/3	$\sqrt{3}/2$	1/2	$\sqrt{3}$	$\sqrt{3}/3$	2	$2\sqrt{3}/3$
90°	π/2	1	0	Undefined	0	Undefined	1
120°	2π/3	$\sqrt{3}/2$	-1/2	$-\sqrt{3}$	$-\sqrt{3}/3$	-2	$2\sqrt{3}/3$
135°	3π/4	$\sqrt{2}/2$	$-\sqrt{2}/2$	-1	-1	$-\sqrt{2}$	$\sqrt{2}$
150°	5π/6	1/2	$-\sqrt{3}/2$	$-\sqrt{3}/3$	$-\sqrt{3}$	$-2\sqrt{3}/3$	2
180°	π	0	-1	0	Undefined	-1	Undefined
210°	7π/6	-1/2	$-\sqrt{3}/2$	$\sqrt{3}/3$	$\sqrt{3}$	$-2\sqrt{3}/3$	-2
225°	5π/4	$-\sqrt{2}/2$	$-\sqrt{2}/2$	1	1	$-\sqrt{2}$	$-\sqrt{2}$
240°	4π/3	$-\sqrt{3}/2$	-1/2	$\sqrt{3}$	$\sqrt{3}/3$	-2	$-2\sqrt{3}/3$
270°	3π/2	-1	0	Undefined	0	Undefined	-1
300°	5π/3	$-\sqrt{3}/2$	1/2	$-\sqrt{3}$	$-\sqrt{3}$	2	$-2\sqrt{3}/3$
315°	7π/4	$-\sqrt{2}/2$	$\sqrt{2}/2$	-1	-1	$\sqrt{2}$	$-\sqrt{2}$
330°	11π/6	-1/2	$\sqrt{3}/2$	$-\sqrt{3}/3$	$-\sqrt{3}$	$2\sqrt{3}/3$	-2
360°	2π	0	1	0	Undefined	1	Undefined

B. Derivatives of Inverse Trigonometric Functions (You must know these!)

1.) $\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$	4.) $\frac{d}{dx}\left(\csc^{-1} x\right) = -\frac{1}{x\sqrt{x^2-1}}$
2.) $\frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1-x^2}}$	5.) $\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{x\sqrt{x^2-1}}$
3.) $\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$	6.) $\frac{d}{dx}\left(\cot^{-1} x\right) = -\frac{1}{1+x^2}$

Examples

1.) Find the exact value of each expression. Your answer should be either a fraction or an integer.

$\cos^{-1}(1) = \square\,\pi$

$\sin^{-1}(\sqrt{2}/2) = \square\,\pi$

$\sin^{-1}(1/2) = \square\,\pi$

$\arctan(\tan(7\pi/6)) =$

$\sin(\arcsin(0.9) =$

2.) Let $f(x) = \left(\tan^{-1} x\right)^7$. Find $f'(x)$.

3.) Let $f(x) = \cos^{-1}\left(e^{8x}\right)$. Find $f'(x)$.

4.) Find the limit: $\lim_{x \to 1^-} \sin^{-1} x =$

5.) Find the limit: $\lim_{x \to \infty} \arctan(-e^{x}) =$

Sec 3.7 Indeterminate forms and L'Hospital's Rule

A. Indeterminate forms

If we have a limit of the form $\lim_{x \to a} \frac{f(x)}{g(x)}$ where both $f(x) \to 0$ and $g(x) \to 0$, then we have the in determinant form of type $\frac{0}{0}$

If we have a limit of the form $\lim_{x \to a} \frac{f(x)}{g(x)}$ where both $f(x) \to \infty$ and $g(x) \to \infty$ then we have the in determinant form of type $\frac{\infty}{\infty}$

B. L'Hospital's Rule

Suppose that $f(x)$ and $g(x)$ are differentiable, $g'(x) \neq 0$ and that $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0}$ or that $\lim_{x \to a} \frac{f(x)}{g(x)} = \pm\frac{\infty}{\infty}$ (i.e. we have an in determinant form of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$), then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Examples:

1.) $\lim_{x \to -2} \frac{x^2 - 4}{x + 2}$

2.) $\lim_{x \to -3} \frac{x^2 - 9}{x + 3}$

3.) $\lim_{x \to 1} \frac{\ln x}{x - 1}$

4.) $\lim_{x \to 0} \frac{\sin x}{1 + \cos x}$

5.) $\lim_{x \to 0} \frac{a^x - 5^x}{9x}$

6.) $\lim_{x \to 0} \frac{1 + x - e^x}{3x^2}$

7.) $\lim_{x \to 0} \frac{1 - e^{ax}}{x^7} =$

8.) $\lim_{x \to 0} \frac{\sin(10x)}{\sin(bx)} =$

9.) $\lim_{x \to 0+} \frac{\ln x}{x^7}$

So the idea is to be able to get your limit problem into the form: $\lim_{x \to a} \frac{f(x)}{g(x)}$ so you can use **L'Hospital's Rule**

If you have $f(x) \cdot g(x)$ and you check to make sure you get either $0 \cdot \infty$ or $\infty \cdot 0$ then you will need to rewrite it first….

you could either rewrite it as $\lim_{x \to a} \frac{f(x)}{1/g(x)}$ *or* $\lim_{x \to a} \frac{g(x)}{1/f(x)}$ *always put the EASY function on the bottom!

10.) $\lim_{x \to 0} \cot 2x \sin 6x$

11.) $\lim_{x \to 0^+} x^4 \ln(x) =$

12.) $\lim_{x \to \infty} x^5 e^{-x^4}$

C. Other "Indeterminate" Forms

$\infty - \infty$ (you will need to rewrite this as either $\frac{0}{0}$ or $\frac{\infty}{\infty}$)

*Try using fractions or factoring

13.) $\lim_{x \to 0} [\csc(ax) - \cot(ax)]$

1.) 0^0 2.) ∞^0 3.) 1^∞	For each of these forms you will need to start by rewriting the problems as y = lim ___. Then you will need to take the **natural log (LN)** of both sides in order to get your exponential function into a multiplication problem using the property of logs. You can then change that into one function divided by another so you can use LH Rule. And lastly, once you get that answer you must set it equal to the **LN y** that you started with on the left hand side. (Phew….It's tough, but you can do it!)

14.) $\lim_{x \to 0^+} x^x$

15.) $\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^{3x} =$

16.) $\lim_{x\to 0}(1-7x)^{\frac{1}{x}} =$