Sec 4.1 Maximum and Minimum Values

A. Absolute Maximum or Minimum / Extreme Values

A function $f(x)$ has an **Absolute Maximum** at $x = c$ if $f(c) \geq f(x)$ for every point x in the domain.

Similarly, $f(x)$ has an **Absolute Minimum** at $x = c$ if $f(c) \leq f(x)$ for every point x in the domain.

B. Local/Relative Maximum or Minimum Values

A function $f(x)$ has a **Local Maximum** at $x = c$ if $f(c) \geq f(x)$ for every point x that is near c.

A function $f(x)$ has a **Local Minimum** at $x = c$ if $f(c) \leq f(x)$ for every point x that is near c.

C. The Extreme Value Theorem

If $f(x)$ is continuous on a closed interval $[a,b]$ then $f(x)$ attains both a maximum and a minimum value on $[a,b]$

D. Fermat's Theorem

If $f(x)$ has a local maximum or minimum at c, and $f'(c)$ exists, then $f'(c) = 0$.

E. Critical Number

A **critical number** of a function $f(x)$, is a number c in the domain such that $f'(c) = 0$ or $f'(c)$ DNE

If $f(x)$ has a local maximum or minimum at c, then c is critical number of $f(x)$

E. Closed Interval Method

To find **Absolute Maximum or Minimum** of a continuous function $f(x)$ on a closed interval $[a,b]$:

1. Find the values of $f(x)$ at the critical numbers of $f(x)$ in (a,b).
2. Find the values of $f(x)$ at the endpoints a and b of the interval.
3. The largest of the values of step 1 and 2 is the **Absolute Maximum**
4. The smallest of the values of step 1 and 2 is the **Absolute Minimum**

Examples:

1.) Find the critical numbers for the following functions

a. $f(x) = 2x^2 + 9x - 2$

b. $f(x) = -2x^3 + 33x^2 - 60x + 11$

c. $f(x) = x^{4/5}(x-5)$

d. $f(x) = (6x-2)e^{-6x}$

2.) Consider the function $f(x) = 3x^2 - 6x + 8, \quad 0 \le x \le 10$.

The absolute maximum value of $f(x)$(on the given interval) is __________
and this occurs at x equals__________

and the absolute minimum of $f(x)$ (on the given interval) is__________
and this occurs at x equals__________

3.) Consider the function $f(x) = 2x^3 + 18x^2 - 162x + 9$ on the interval $-9 \le x \le 4$.
Find the critical numbers and absolute minimum and maximum values.

4.) Consider the function $f(x) = x + 2\cos x$ on the interval $0 \le x \le \pi$. Find the absolute maximum and minimum of the function.

5.) Choose the best reason that the function $f(x) = x^{91} + x^{25} + x^{7} + 13$ has neither a local maximum nor a local minimum.

(a) The function f(x) is always positive.
(b) The derivative f ' (x) is always negative.
(c) The derivative f ' (x) is always positive.
(d) The highest power of x in f(x) is odd.

Sec 4.2 The Mean Value Theorem

A. Rolle's Theorem

Let $f(x)$ be a function such that $f(x)$ is continuous on $[a,b]$, $f(x)$ is differentiable on (a,b) and $f(a)=f(b)$
Then there is a number c in (a,b) such that $f'(c)=0$

B. The Mean Value Theorem

Let $f(x)$ be a function such that $f(x)$ is continuous on $[a,b]$ and $f(x)$ is differentiable on (a,b) .
Then there is a number c in (a,b) such that $f'(c)=\dfrac{f(b)-f(a)}{b-a}$
Or equivalently $f(b)-f(a)=f'(c)\cdot(b-a)$

C. Constant Theorem

If $f'(x)=0$ for all x in an interval (a,b) then $f(x)$ constant on (a,b).

D. Corollary

If $f'(x)=g'(x)$ for all x in an interval (a,b) then $f-g$ is constant on (a,b) (i.e. $f(x)=g(x)+c$)

Examples

1.) Consider the function $f(x)=x^2-4x+1$ on the interval $[0,4]$. Verify that this function satisfies the three hypotheses of Rolle's Theorem on the interval.

$f(x)$ is_______________on $[0,4]$; $f(x)$ is________________on $(0,4)$; and $f(0)=f(4)=$____________

Then by Rolle's theorem, there exists a c such that $f'(c)=0$. Find the value c.

2.) Consider the function $f(x)=3x^2+5x+11$ on the interval $[-3,6]$. Find the average or mean slope of the function on this interval, i.e.

$$\frac{f(6) - f(-3)}{6 - (-3)} =$$

By the Mean Value Theorem, we know there exists a c in the open interval $(-3, 6)$ such that $f'(c)$ is equal to this mean slope. For this problem, there is only one c that works. $c =$

3.) By applying Rolle's Theorem, check whether it is possible that the function $f(x) = x^5 + x - 11$ has two real roots.

Possible or impossible?

Your reason is that if $f(x)$ has two real roots then by Rolle's theorem: $f'(x)$ must be ___________

at certain value of x between these two roots, but $f'(x)$ is always negative ,positive, or zero______________

4.) Suppose $f(x)$ is continuous on [2,8] and $2 \leq f'(x) \leq 8$ for all x in (2,8) . Use the Mean Value Theorem to estimate $f(8) - f(2)$.

_____ $\leq f(8) - f(2) \leq$ _____

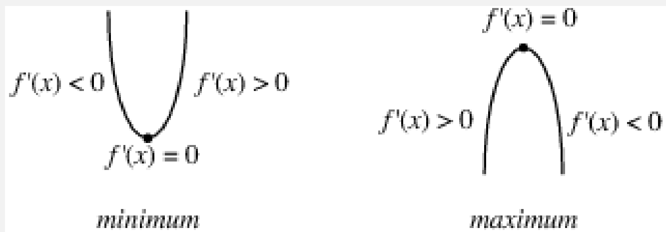
A. The First Derivative

Increasing/Decreasing Test

- If $f'(x) > 0$ on an interval, then $f(x)$ is increasing on that interval
- If $f'(x) < 0$ on an interval, then $f(x)$ is decreasing on that interval

A **critical number** of a function $f(x)$, is a number c in the domain such that $f'(c) = 0$ or $f'(c)$ DNE

If $f(x)$ has a local maximum or minimum at c, then c is critical number of $f(x)$



The First Derivative Test

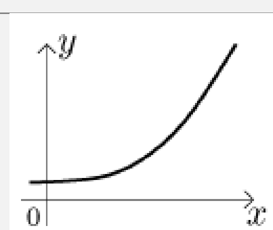
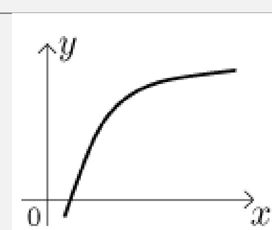
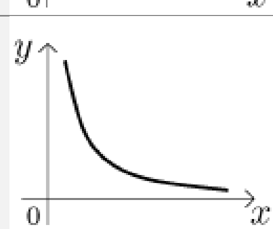
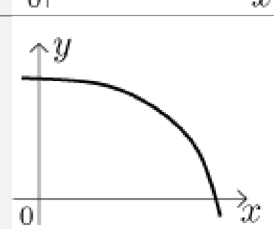
Suppose c is a critical number of a continuous function $f(x)$

- If $f'(x)$ changes from positive to negative at c, then $f(x)$ has a local maximum at c
- If $f'(x)$ changes from negative to positive at c, then $f(x)$ has a local minimum at c
- If $f'(x)$ does not change sign at c, then $f(x)$ has no local maximum or minimum at c

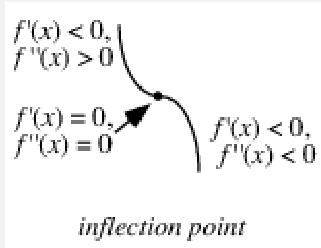
B. The Second Derivative

Concavity

- If $f''(x) > 0$ on an interval, then $f(x)$ is concave up on that interval
- If $f''(x) < 0$ on an interval, then $f(x)$ is concave down on that interval

	Concave Up	Concave Down
Increasing Slope		
Decreasing Slope		

An **inflection point** of a function $f(x)$, is a point at which the curvature (second derivative) changes sign. The curve changes from being concave upwards (positive curvature) to concave downwards (negative curvature), or vice versa.



The Second Derivative Test

Suppose that $f''(x)$ is continuous at c

- If $f'(x)=0$ and $f''(x)>0$ then $f(x)$ has a local minimum at c.
- If $f'(x)=0$ and $f''(x)<0$ then $f(x)$ has a local maximum at c.

Example:

1.) $f(x)=2x^3-3x^2-12x$

a.) Find the critical points and the intervals on increase and decrease.
b.) State whether each critical point is a maximum or a minimum.
c.) Find the inflection points and the intervals on concavity.
d.) Sketch the graph and verify your results.

2.) $g(x) = x + 2\cos(x)$ on $0 \le x \le 2\pi$

a.) Find the critical points and the intervals on increase and decrease.
b.) State whether each critical point is a maximum or a minimum.
c.) Find the inflection points and the intervals on concavity.
d.) Sketch the graph and verify your results.

3.) $h(x) = \frac{e^x}{e^x + 8}$

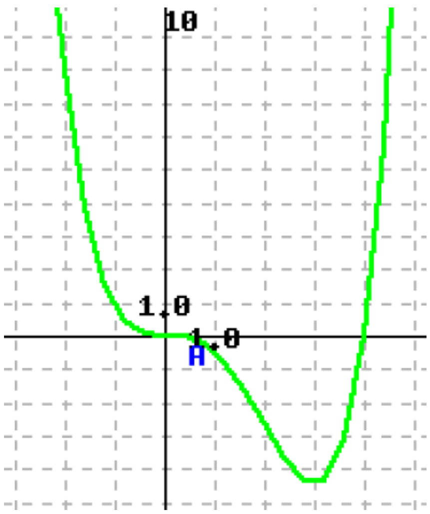
a.) Find the critical points and the intervals on increase and decrease.
b.) State whether each critical point is a maximum or a minimum.
c.) Find the inflection points and the intervals on concavity.
d.) Sketch the graph and verify your results..

4.) Suppose that $f''(x)$ is continuous on $(-\infty,\infty)$.

a.) If $f'(5)=0$ and $f''(5)=6$, then f has a local_______________ at $x=5$.

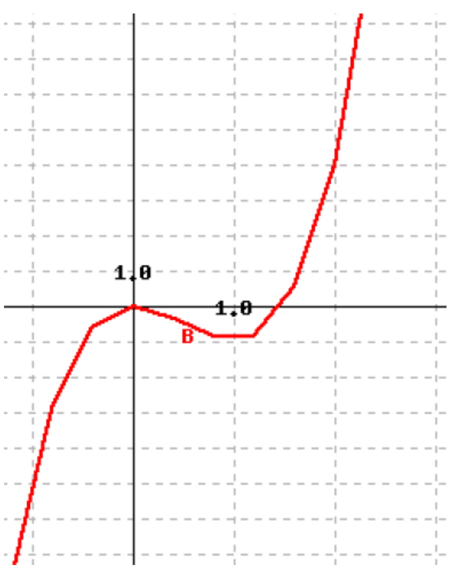
b.) If $f'(19)=0$ and $f''(19)=-6$, then f has a local_______________ at $x=19$.

5.) Given the graph of $f(x)$, determine whether the following conditions are true.



1. $f'(3)=0$
2. $f''(x)>0$ if $0<x<2$
3. $f''(x)>0$ if $x<0$
4. $f'(x)\le 0$ if $x<3$
5. $f'(0)=0$

6.) Given the graph of $f'(x)$, determine whether the following conditions are true.



1. f is concave downward on the interval $(-\infty,0)$
2. f has a local maximum at x = 0
3. f is decreasing on the interval $(-\infty,1.5)$
4. f has an inflection point at x = 0
5. f is decreasing on the interval $(0,1)$
6. f is increasing on the interval $(1.5,\infty)$

7.) Find a cubic function $f(x) = ax^3 + cx^2 + d$ that has a local maximum value of 8 at $x = -2$ and a local minimum value of 6 at $x = 0$.

Sec 4.4 Curve Sketching

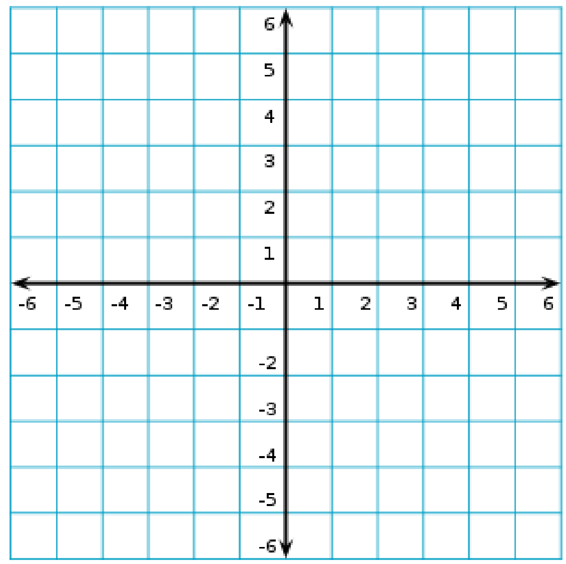
A. Guidelines for sketching a curve

1. Domain
2. Intercepts (x-intercepts and y-intercepts)
3. Symmetry (Odd, even or periodic functions)
4. Asymptotes
5. Intervals of Increase and Decrease
6. Maximum and Minimum Values
7. Intervals of Concavity

Example: Sketch the curve using the guidelines 1 – 7.

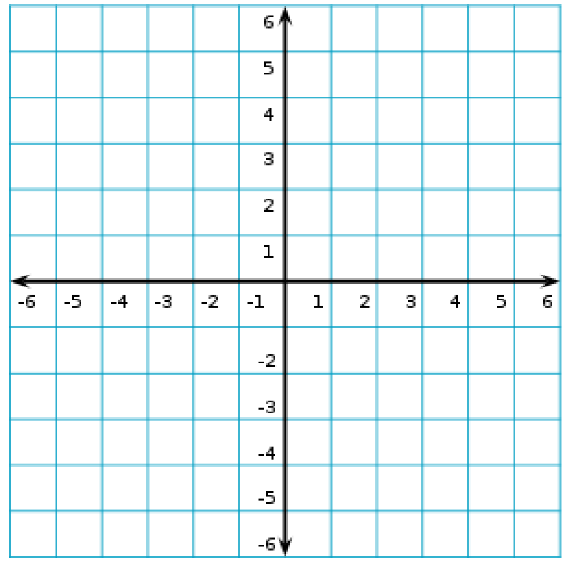
1.) $f(x)=\dfrac{x^2}{x^2-4}$

1. Domain
2. Intercepts (x-intercepts and y-intercepts)
3. Asymptotes
4. Intervals of Increase and Decrease
5. Maximum and Minimum Values
6. Intervals of Concavity
7. Inflection Points



2.) $f(x) = 3\cos(x) - \cos^3(x)$ on $0 \le x \le 2\pi$

1. Domain
2. Intercepts (x-intercepts and y-intercepts)
3. Asymptotes
4. Intervals of Increase and Decrease
5. Maximum and Minimum Values
6. Intervals of Concavity
7. Inflection Points



B. Guidelines for sketching a function given a sketch of it's derivative.

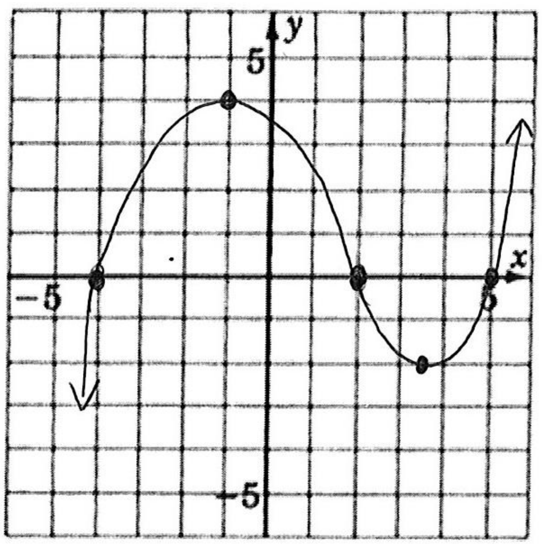
1. Find all intervals where the function is increasing and decreasing
2. Find all intervals where the function is concave up and concave down
3. Sketch a function that has these characteristics (there are many graphs possible)

f(x) : max and mins ↔ f '(x): roots	
f(x): inflection points ↔ f '(x): max and mins	
f(x): concave up ↔ f '(x): increasing	f(x): concave down ↔ f '(x): decreasing
f(x): increasing ↔ f '(x) > 0	f(x): decreasing ↔ f '(x) < 0

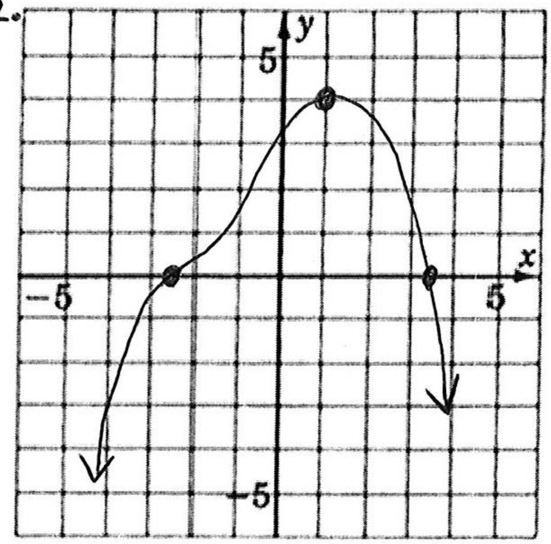
For each of the following state where *f(x)* is increasing, decreasing, has max and mins, concave up and down, inflection points and sketch *f(x)*.

1.

f '(x)

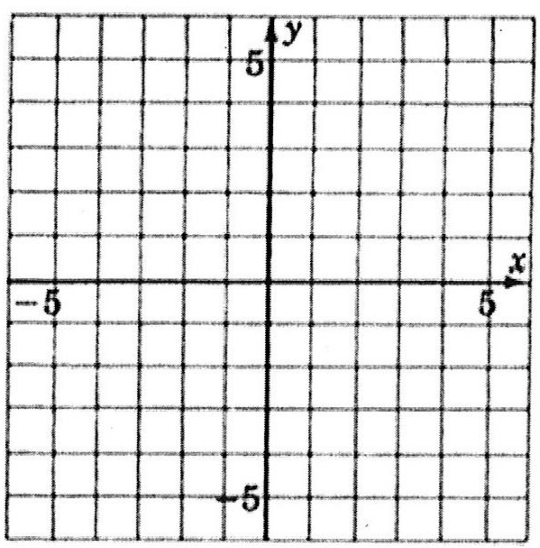


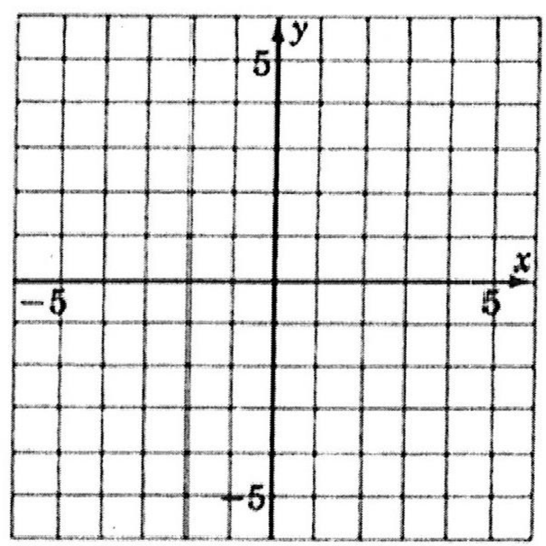
2.



f '(x)

f(x)

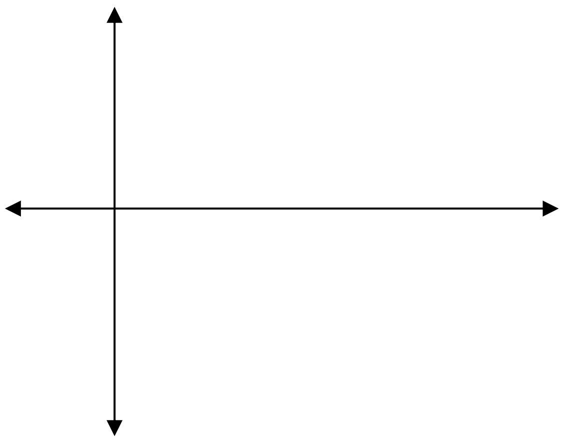




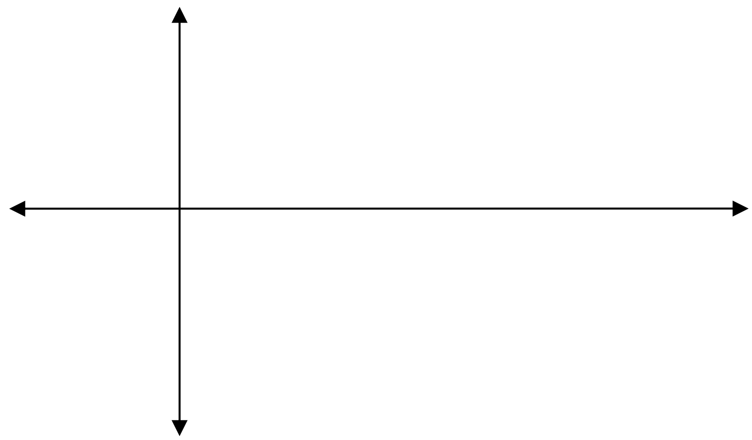
f(x)

Sketch the graph of a function, *f(x)*, that satisfies all of the given conditions.

3. $f'(0) = f'(2) = f'(4) = 0$
$f'(x) > 0$ if $x < 0$ or $2 < x < 4$
$f'(x) < 0$ if $0 < x < 2$ or $x > 4$
$f''(x) > 0$ if $1 < x < 3$
$f''(x) < 0$ if $x < 1$ or $x > 3$



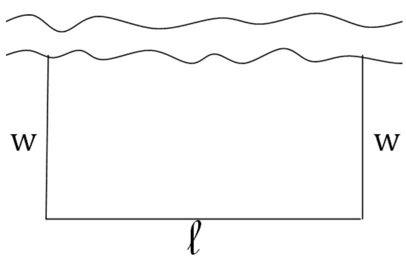
4. $f(0) = 0$, $f'(-2) = f'(1) = f'(9) = 0$
$\lim_{x \to \infty} f(x) = 0$ $\lim_{x \to 6} f(x) = -\infty$
$f'(x) < 0$ on $(-\infty,-2)$, $(1,6)$ and $(9,\infty)$
$f'(x) > 0$ on $(-2,1)$ and $(6,9)$
$f''(x) > 0$ on $(-\infty,0)$ and $(12,\infty)$
$f''(x) < 0$ on $(0,6)$ and $(6,12)$



Sec 4.5 Optimization

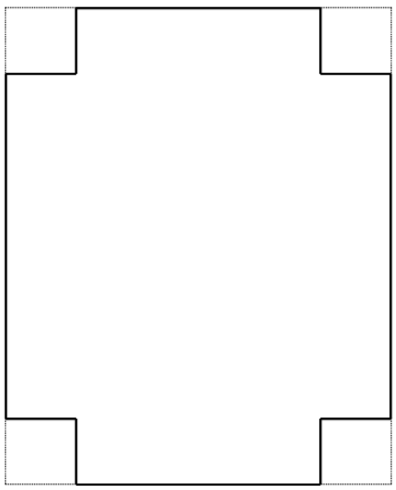
Examples

1.) Farmer Brown has 1200 ft of fence to create a rectangular pen that will be adjacent to a river. If he does not need to put any fence on the side that borders the river, what dimensions will maximize the area of the pen, and what is the maximum area? (Do not forget units!)

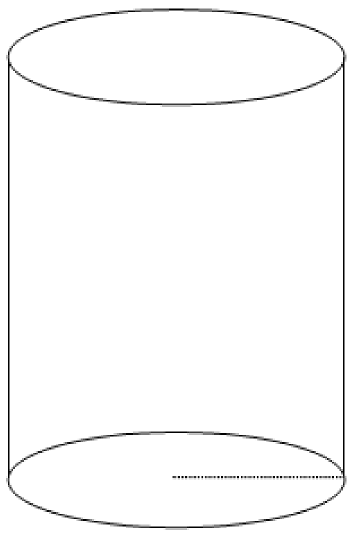


2.) Find two numbers A and B (with $A \leq B$) whose difference is 42 and whose product is minimized.

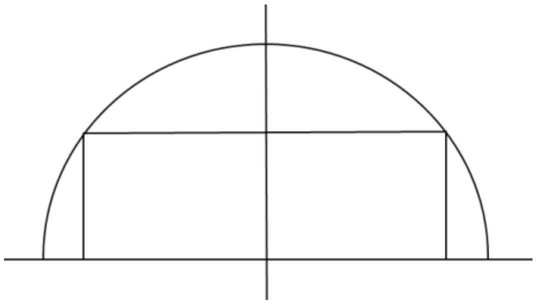
3.) A box is to be made out of a 10 by 18 piece of cardboard. Squares of equal size will be cut out of each corner, and then the ends and sides will be folded up to form a box with an open top. Find the length L, width W, and height H of the resulting box that maximizes the volume. (Assume that $W \leq L$).



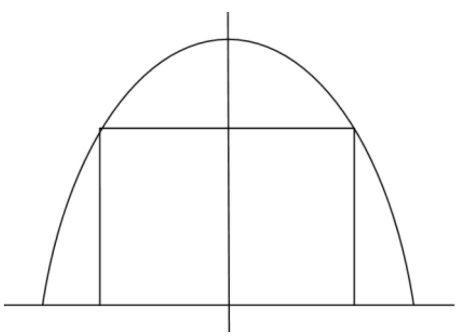
4.) A cylindrical oatmeal container has a capacity of 3 liters. Find the dimensions that will minimize the cost of production material to construct the container.



5.) Find the area of the largest rectangle that can be inscribed in a semicircle with a radius 4

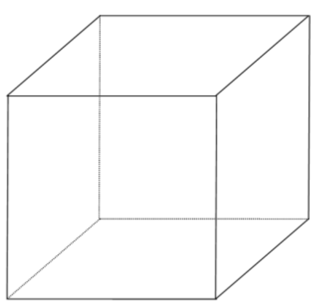


6.) Find the dimensions of the rectangle of largest area that has its base on the x-axis and its other two vertices above the x-axis and lying on the parabola $y = 4 - x^2$.



7.) Find the point on the line $y = 4x + 7$ which is closest to the point $(0,0)$.

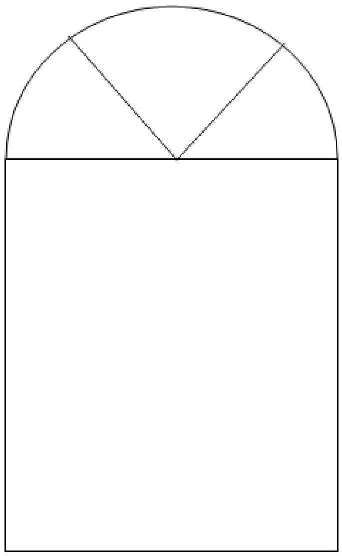
8.) If 2000 square centimeters of material is available to make a box with a square base and an open top, find the largest possible volume of the box.



9.) A piece of wire 12 m long is cut into two pieces. One piece is bent into the shape of a circle of radius r and the other is bent into a square of side s. How should the wire be cut so that the total area enclosed is:

a.) Maximized
b.) Minimized

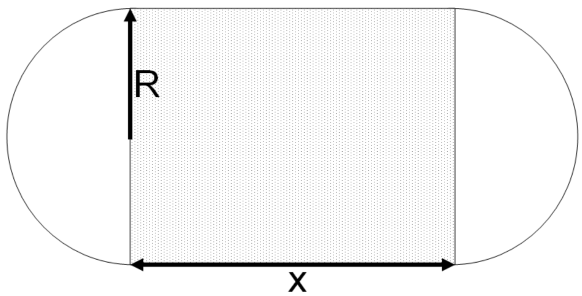
10.) A Norman window has the shape of a semicircle atop a rectangle so that the diameter of the semicircle is equal to the width of the rectangle. What is the area of the largest possible Norman window with a perimeter of 45 feet?



11.) A running track has the shape of a rectangle with a semicircle on each end. If the length of the track is 400 meters, find the dimensions so that

a.) the rectangular (shaded)region is maximized.

b.) The entire region is maximized.



Sec 4.6 Newton's Method

A. The Newton's Method Formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Examples:

1.) Starting with $x_0 = 2$ find the third approximation x_3 to the root of the equation $x^3 - 2x - 5 = 0$

n	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
0				
1				
2				
3				

2.) Starting with $x_0 = 1$ find the third approximation x_3 to the root of the equation $\tan^{-1}(x) = 1 - x$

n	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
0				
1				
2				
3				

To find these approximations using the calculator:
Let Y1 = *f(x)* and let Y2 = *f '(x)*
Then in the HOME SCREEN type in x_0 and press ENTER
Type in immediately after you hit ENTER: – Y1(Ans) / Y2(Ans) and press ENTER
(Each time you press enter you will get the next approximation of the root.)

3. a.) Find the equation $f(x)$ that results in a solution of $\sqrt[4]{9}$

b.) Find the second, third and fourth approximations of the root to this function if $x_0 = 2$

4.) Find the fourth approximation x_2 to the root of the equation $e^{-x} = 2 + x$

n	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
0				
1				
2				

Sec 4.7 Anti Derivatives

A function $F(x)$ is called the anti-derivative of $f(x)$ if $F'(x) = f(x)$

Basic rules of anti differentiation

In general: Reverse basic rules of differentiation.	
$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C,$ when $n \neq -1$	$\int \cos x\,dx = \sin x + C$
$\int e^x\,dx = e^x + C$	$\int \sin x\,dx = -\cos x + C$
$\int a^x\,dx = \frac{a^x}{\ln a} + C$	$\int \sec^2 x\,dx = \tan x + C$
$\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$	$\int \csc^2 x\,dx = -\cot x + C$
$\int \frac{1}{1+x^2}\,dx = \arctan x + C$	$\int \sec x \tan x\,dx = \sec x + C$
$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$	$\int \csc x \cot x\,dx = -\csc x + C$

Important: **Always use proper notation!*
**Don't forget +C*

Examples: Find the anti derivative for each of the following:

1.) $f(x) = 2x^3 - 7x^2 + 7x - 7$

2.) $g(x) = 6x^{\frac{1}{5}} - 18x^{\frac{4}{5}}$

3.) $h(x) = \sqrt[7]{x^2} + \frac{2}{x^5} + \frac{1}{x}$

4.) $k(x) = \frac{x^4 + 8\sqrt{x}}{x^2}$

5.) $m(x) = 10\sin x + 6\cos x$.

6.) $k(x) = \frac{8}{1+x^2}$

7.) $k(x) = \frac{2}{\sqrt{1-x^2}}$

8.) Find the function $F(x)$ given that $f(x) = 2x^7 - 4x^3$ and $F(0) = 19$

9.) Find the function $f(x)$ given that $f''(x) = 24x^2 + 10$, $f(0) = 5$ and $f'(1) = 2$

10.) A particle is moving with acceleration $a(t) = 12t + 2$. Its position at time $t = 0$ is $s(0) = 11$ and its velocity at time $t = 0$ is $v(0) = 9$. What is its position at time $t = 6$?