Sec 2.6 Differentiation of Implicit Functions

Recall from the previous week, that when we take the derivative of $y = f(x)$ then $y' = f'(x)$ where $f(x)$ is a function in terms of x (i.e. the only variable in the function is x)

Example: If $y = x^3 + 2x$ then $y' = \frac{dy}{dx} = 3x^2 + 2$

So, in other words, to take a derivative this way, we have to have the equation solved for "y".

Example: If $x = x^3 + 2x - y$. Here we first have to solve for y. So $y = x^3 + 2x - x \quad \Rightarrow \quad y = x^3 + x$

then $y' = \frac{dy}{dx} = 3x^2 + 1$

Example: If $yx = x^3 + 2xy^2 - y$. Again, we have to solve for y in order to take a derivative in the way that we have learnt in the preceding chapters. However, (as you can see in this case) it is not always easy/possible to do so.

HENCE: Implicit Differentiation!

A. Implicit Differentiation

Implicit Differentiation: Differentiation of a function where one variable (typically y) is not explicitly expressed a s a function of another variable (typically x).

<u>Here's how it works:</u>

- It is important to pay attention to the notation. If we are given an equation in terms of x and y, and asked to find y' or $\frac{dy}{dx}$, we need to see that we are finding the derivative of **y**, with respect to **x**.
- We will treat both x and y like a variable, and take derivatives of each, but;
- When we take a derivative of a term containing "x" we will proceed as usual
- When we take a derivative of a term containing "y" we will proceed as usual AND then also multiply the derivative of that term by $\frac{dy}{dx}$ (or y').
- We will use product, quotient and chain rules as needed.
- After differentiating, solve for (i.e. isolate) y' or $\frac{dy}{dx}$,

Example: Find $\frac{dy}{dx}$ for $x = x^3 + y^2$.

$$\frac{d(x)}{dx} = \frac{d(x^3)}{dx} + \frac{d(y^2)}{dx} \quad \Rightarrow \quad 1 = 3x^2 + 2y \cdot \frac{dy}{dx}$$

Since we are trying to find $\frac{dy}{dx}$, isolate $\frac{dy}{dx}$ in our equation: $1 - 3x^2 = 2y \cdot \frac{dy}{dx} \quad \Rightarrow \quad \frac{1-3x^2}{2y} = \frac{dy}{dx}$

Example: Find $\frac{dy}{dx}$ for $x = 4x^3 + y^2 - 8y$.

$$\frac{d(x)}{dx} = \frac{d(4x^3)}{dx} + \frac{d(y^2)}{dx} - \frac{d(8y)}{dx} \quad \Rightarrow \quad 1 = 12x^2 + 2y \cdot \frac{dy}{dx} - 8 \cdot \frac{dy}{dx}$$

Since we are trying to find $\frac{dy}{dx}$, isolate $\frac{dy}{dx}$ in our equation:

$$1 - 12x^2 = 2y \cdot \frac{dy}{dx} - 8 \cdot \frac{dy}{dx} \quad \Rightarrow \quad 1 - 12x^2 = \frac{dy}{dx}(2y - 8) \quad \Rightarrow \quad \frac{1 - 12x^2}{2y - 8} = \frac{dy}{dx}$$

Example: Find y' for $x = x^3 y^2 - 3y^3$. (Notice that in this problem we have $x^3 y^2$ - a product of x and y. Here we will have to use the product rule.)

$$\frac{d(x)}{dx} = \frac{d(x^3 y^2)}{dx} - \frac{d(3y^3)}{dx} \quad \Rightarrow \quad 1 = [(3x^2)(y^2) + (x^3)(2y \cdot y')] - 9y^2 \cdot y'$$

$$\Rightarrow \quad 1 - (3x^2)(y^2) = (x^3 \cdot 2y) \cdot y' - (9y^2) \cdot y' \quad \Rightarrow \quad 1 - (3x^2)(y^2) = y'(x^3 \cdot 2y - 9y^2)$$

$$\Rightarrow \quad \frac{1 - 3x^2 y^2}{2x^3 y - 9y^2} = y'$$

Examples: Find $\frac{dy}{dx}$ for the following:

1.) $x^2 + 2y^2 - 11 = 0$

2.) $y^2x-\frac{5y}{x+1}+3x=4$

3.) Find y' for $2y+5-x^2-y^3=0$ and evaluate at $(2,-1)$

4.) Find $\frac{dA}{dt}$ for $A=\pi r^2$

5.) Find $\frac{dV}{dt}$ for $V = \frac{1}{3}\pi r^2 h$

6.) If $f(x) + x^7[f(x)]^3 = 11$ and $f(2) = 6$, find $f'(2) =$

7.) Use implicit differentiation to find an equation of the tangent line to the curve
line to the curve $4x^2 - 4xy - 1y^3 = 84$ at the point $(1, -4)$ of the form $y = mx + b$
$m =$ _____ and $b =$ _____