Key

Homework Set 18

(sect 8.5: Power Series)

Find the radius of convergence and the interval of convergence for each power series.

$$\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{n+1}$$

$$\lim_{n\to\infty} \left| \frac{(-3)^{n+1} x^{n+1}}{n+2} \cdot \frac{n+1}{(-3)^n x^n} \right| = \lim_{n\to\infty} \left| \frac{-3x(n+1)}{n+2} \right| = 3|x| \lim_{n\to\infty} \frac{n+1}{n+2} = 3|x| < 1$$

$$\Downarrow$$

$$|x| < \frac{1}{3}$$

$$\Downarrow$$

$$-\frac{1}{3} < x < \frac{1}{3}$$

$x = \frac{1}{3} \Rightarrow \sum \frac{(-3)^n (\frac{1}{3})^n}{n+1} = \sum \frac{(-1)^n}{n+1}$ Conv. (alt series)

$x = -\frac{1}{3} \Rightarrow \sum \frac{(-3)^n (-\frac{1}{3})^n}{n+1} = \sum \frac{1}{n+1}$ Diver. (compare with $\sum \frac{1}{n}$)

$R = \frac{1}{3}$

IC: $(-\frac{1}{3}, \frac{1}{3}]$ or $-\frac{1}{3} < x \leq \frac{1}{3}$

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n 4^n}$$

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{n+1}}{(n+1) 4^{n+1}} \cdot \frac{n 4^n}{(-1)^n x^n} \right| = \lim_{n\to\infty} \frac{|x| \cdot n}{4(n+1)} = \frac{|x|}{4} < 1$$

$R = 4$

$|x| < 4 \Rightarrow -4 < x < 4$

IC: $(-4, 4]$ or $-4 < x \leq 4$

$x = -4$: $\sum \frac{(-1)^n (-4)^n}{n 4^n} = \sum \frac{1}{n}$ diverges (harmonic)

$x = 4$: $\sum \frac{(-1)^n 4^n}{n 4^n} = \sum \frac{(-1)^n}{n}$ converges (alt series)

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n\to\infty} \frac{|x|}{n+1} = 0$$ converges everywhere

IC: $(-\infty, \infty)$ or $\mathbb{R}$ and $R = \infty$

$$\sum_{n=0}^{\infty} \frac{(-1)^n n! (x+7)^n}{2^n}$$

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1} (n+1)! (x+7)^{n+1}}{2^{n+1}} \cdot \frac{2^n}{(-1)^n n! (x+7)^n} \right| = \lim_{n\to\infty} \frac{(n+1)|x+7|}{2} = \infty$$

IC: $x = -7$ or $\{-7\}$

$R = 0$

$$\sum_{n=1}^{\infty} \sqrt{n}(2x-4)^n = \sum_{n=0}^{\infty} \sqrt{n} \cdot 2^n \cdot (x-2)^n$$

$$\lim_{n\to\infty} \left| \frac{\sqrt{n+1} \cdot 2^{n+1} (x-2)^{n+1}}{\sqrt{n} \cdot 2^n (x-2)^n} \right| = 2|x-2| \cdot \lim_{n\to\infty} \frac{\sqrt{n+1}}{\sqrt{n}} = 2|x-2| < 1$$

$$\Rightarrow |x-2| < 1/2 \Rightarrow -\tfrac{1}{2} < x-2 < 1/2 \Rightarrow \tfrac{3}{2} < x < 5/2$$

$R = 1/2$

IC: $(\frac{3}{2}, 5/2)$

$x = 3/2$: $\sum \sqrt{n}(3-4)^n = \sum \sqrt{n}(-1)^n$

Diverges

Alt series

$x = 5/2$: $\sum \sqrt{n}(1)^n = \sum \sqrt{n}$

Diverges

test for divergence

$$\sum_{n=1}^{\infty} \frac{12(x+2)^n}{(-5)^n \sqrt{n}}$$

$$\lim_{n\to\infty} \left| \frac{12(x+2)^{n+1}}{(-5)^{n+1}\sqrt{n+1}} \cdot \frac{(-5)^n \sqrt{n}}{12(x+2)^n} \right| = \lim_{n\to\infty} \frac{|x+2|\sqrt{n}}{5\sqrt{n+1}} = \frac{|x+2|}{5} < 1$$

$$\Rightarrow |x+2| < 5 \Rightarrow -5 < x+2 < 5 \Rightarrow -7 < x < 3$$

$R = 5$

IC: $[-7, 3)$

$x = -7$: $\sum \frac{12(-5)^n}{5^n \sqrt{n}} = \sum \frac{12(-1)^n}{\sqrt{n}}$

converges

alt series

$x = 3$: $\sum \frac{12(5)^n}{5^n\sqrt{n}} = \sum \frac{12}{\sqrt{n}}$

$p = 1/2$

Diverges

$$\sum_{n=1}^{\infty} \frac{n(x-1)^n}{n^2+1}$$

$$\lim_{n\to\infty} \left| \frac{(n+1)(x-1)^{n+1}}{(n+1)^2+1} \cdot \frac{n^2+1}{n(x-1)^n} \right| = \lim_{n\to\infty} \frac{|x-1| \cdot (n^3+n^2+n+1)}{n^3+2n^2+2n} = |x-1| < 1$$

$$\Rightarrow -1 < x-1 < 1 \Rightarrow 0 < x < 2$$

$R = 1$

IC: $[0, 2)$

$x = 0$: $\sum \frac{n(-1)^n}{n^2+1}$

Converges

alt series

$x = 2$: $\sum \frac{n(1)^n}{n^2+1} = \sum \frac{n}{n^2+1}$

diverges

compare to $\sum \frac{1}{n}$

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!\,(n+1)!\,2^{2n+1}}$$

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2(n+1)+1}}{(n+1)!\,(n+2)!\,2^{2(n+1)+1}} \cdot \frac{n!\,(n+1)!\,2^{2n+1}}{(-1)^n x^{2n+1}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x^{2n+3} \cdot n! \cdot 2^{2n+1}}{(n+2)! \cdot 2^{2n+3} \cdot x^{2n+1}} \right| = \lim_{n\to\infty} \frac{x^2}{(n+2)(n+1) \cdot 2^2} = 0$$

Converges everywher

$R = \infty$

IC: $(-\infty, \infty)$ or $\mathbb{R}$