$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots$

Key

Homework Set 20

(sect 8.7: Taylor & Binomial Series)

Find the Taylor series representation for each function centered at the given value of a.

1. $f(x) = 2 - 3x + x^2,\ a = -1$

$$= 6 + \frac{-5}{1!}(x+1) + \frac{2}{2!}(x+1)^2$$
$$= 6 - 5(x+1) + (x+1)^2$$

$f(x) = 2-3x+x^2$	$f(-1) = 6$
$f'(x) = -3+2x$	$f'(-1) = -5$
$f''(x) = 2$	$f''(-1) = 2$
$f'''(x) = 0$	$f'''(-1) = 0$

2. $f(x) = e^x,\ a = 5$

$$= e^5 + \frac{e^5}{1!}(x-5) + \frac{e^5}{2!}(x-5)^2 + \ldots + \frac{e^5}{n!}(x-5)^n + \ldots$$
$$= \sum_{n=0}^{\infty} \frac{e^5}{n!}(x-5)^n$$

$f(x) = e^x$	$f(5) = e^5$
$f'(x) = e^x$	$f'(5) = e^5$
$\vdots$	$\vdots$
$f^{(n)}(x) = e^x$	$f^{(n)}(5) = e^5$

3. $f(x) = \cos x,\ a = \frac{\pi}{2}$

$$= 0 + \frac{-1}{1!}(x-\pi/2) + 0 + \frac{1}{3!}(x-\pi/2)^3 + 0 + \frac{-1}{5!}(x-\pi/2)^5 + \ldots$$
$$= \sum_{n=1}^{\infty} \frac{(-1)^n (x-\pi/2)^{2n-1}}{(2n-1)!}$$

$f(x) = \cos x$	$f(\pi/2) = 0$
$f'(x) = -\sin x$	$f'(\pi/2) = -1$
$f''(x) = -\cos x$	$f''(\pi/2) = 0$
$f'''(x) = \sin x$	$f'''(\pi/2) = 1$
$f^{(4)}(x) = \cos x$	$f''''(\pi/2) = 0$

4. $f(x) = \ln x,\ a = 2$

$$= \ln 2 + \frac{1}{2}(x-2) + \frac{-1/4}{2}(x-2)^2 + \frac{2/8}{6}(x-2)^3 + \ldots$$
$$= \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!}{2^n}(x-2)^n$$

$f(x) = \ln x$	$f(2) = \ln 2$
$f'(x) = 1/x$	$f'(2) = 1/2$
$f''(x) = -1/x^2$	$\vdots$
$f'''(x) = 2/x^3$	$f^{(n)}(2) = \frac{(-1)^{n-1}(n-1)!}{2^n}$
$f^{(4)}(x) = -6/x^4$	
$\vdots$	
$f^{(n)}(x) = \frac{(n-1)!(-1)^{n-1}}{x^n}$	

Use a known Maclaurin series to evaluate each of the following expressions:

5. Obtain the Maclaurin series for $2x\sin(x^3)$.

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$
$$2x\sin(x^3) = 2x^4 - \frac{2x(x^3)^3}{3!} + \frac{2x(x^3)^5}{5!} - \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{2x^{6n+4}}{(2n+1)!}$$

6. Evaluate the indefinite integral as an infinite series: $\int \frac{e^x - 1}{x}\,dx$.

$$\int \frac{e^x - 1}{x}\,dx = \int \frac{\left[1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots\right] - 1}{x}\,dx = \int \left(1 + \frac{1}{2!}x + \frac{1}{3!}x^2 + \frac{1}{4!}x^3 + \ldots\right)dx$$
$$= x + \frac{1}{2\cdot 2!}x^2 + \frac{1}{3\cdot 3!}x^3 + \frac{1}{4\cdot 4!}x^4 + \ldots = \sum_{n=1}^{\infty} \frac{1}{n\cdot n!}x^n$$

7. Evaluate the indefinite integral as an infinite series: $\int \tan^{-1}(x^2)\,dx$.

$$\int \tan^{-1}(x^2)\,dx = \int (x^2) - \frac{(x^2)^3}{3} + \frac{(x^2)^5}{5} - \frac{(x^2)^7}{7} + \ldots\,dx = \frac{1}{3}x^3 - \frac{x^7}{3\cdot 7} + \frac{x^{11}}{5\cdot 11} - \frac{x^{15}}{7\cdot 15} + \ldots$$
$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+3}}{(2n+1)(4n+3)}$$

8. Compute: $\lim_{x\to 0} \frac{\tan x - x}{x^3}$

$\lim_{x\to 0}$ [

~~9. Use the Maclaurin series for $\sin x$ to evaluate $\sin 5$ correct to 4 decimal places.~~

~~10. Use series to approximate $\int_0^{0.5} x^2 e^{-x^2}\, dx$ to within 0.001.~~

11.

a. Expand $f(x) = \frac{x}{(1-x)^2}$ as a power series.

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \dots = \sum_{n=0}^{\infty} (n+1)x^n \qquad \text{(see hw set 19, \# 4)}$$

$$\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + \dots = \sum_{n=1}^{\infty} n x^n$$

b. Use part (a) to the sum of the series $\sum_{n=1}^{\infty} \frac{n}{2^n}$

$$\sum_{n=1}^{\infty} n\left(\tfrac{1}{2}\right)^n = f(1/2)$$

$$= \frac{1/2}{(1-1/2)^2} = \frac{1/2}{1/4} = 2$$

12. Find the sum of the series: $\sum_{n=0}^{\infty} \frac{3^n}{5^n n!}$

$$\sum_{n=0}^{\infty} \frac{3^n}{5^n n!} = \sum_{n=0}^{\infty} \frac{(3/5)^n}{n!} \qquad \text{use} \quad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$= e^{3/5}$$

Use the Binomial series to expand the given function as a power series.

13. $f(x) = \frac{1}{(1+x)^3} = (1+x)^{-3} = \sum_{n=0}^{\infty} \binom{-3}{n} x^n \quad , k = -3$

$$= 1 - 3x + \frac{-3(-4)}{2!}x^2 + \frac{-3(-4)(-5)}{3!}x^3 + \frac{-3(-4)(-5)(-6)}{4!}x^4 + \dots$$

$$= 1 - 3x + 6x^2 - 10x^3 + 15x^4 + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{(n+1)(n+2)}{2} x^n$$

14. $f(x) = (1-x)^{1/5} = \sum_{n=0}^{\infty} \binom{1/5}{n}(-x)^n \quad , k = 1/5$

$$= 1 - \tfrac{1}{5}x + \frac{\frac{1}{5}(-\frac{4}{5})}{2!}x^2 - \frac{\frac{1}{5}(-\frac{4}{5})(-\frac{9}{5})}{3!}x^3 + \frac{\frac{1}{5}(-\frac{4}{5})(-\frac{9}{5})(-\frac{14}{5})}{4!}x^4 + \dots$$

$$= 1 - \tfrac{1}{5}x - \frac{2}{25}x^2 - \frac{6}{125}x^3 - \frac{21}{625}x^4 - \dots$$

15. $f(x) = \frac{x^2}{\sqrt{9-x}} = x^2(9-x)^{-1/2} = \frac{1}{3}x^2(1-x/9)^{-1/2} = \frac{1}{3}x^2(1+(-x/9))^{-1/2} = \frac{1}{3}x^2 \sum_{n=0}^{\infty} \binom{-1/2}{n}(-x/9)^n \quad , k = -\frac{1}{2}$

$$= \tfrac{1}{3}x^2\left[1 - \tfrac{1}{2}(-x/9) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}(-x/9)^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!}(-x/9)^3 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})(-\frac{7}{2})}{4!}\left(-\tfrac{x}{9}\right)^4 + \dots\right]$$

$$= \tfrac{1}{3}x^2\left[1 + \tfrac{1}{18}x + \frac{1}{216}x^2 + \frac{5}{11664}x^3 + \frac{35}{52488}x^4 + \dots\right]$$

$$= \tfrac{1}{3}x^2 + \tfrac{1}{54}x^3 + \frac{1}{648}x^4 + \frac{5}{[illegible]}x^5 + \frac{35}{[illegible]}x^6 + \dots$$