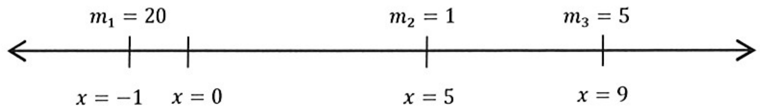
Homework Set 25

(sect 7.5: Centroids)

1. Point-masses m_i are located on the x-axis (see below). Find the moment M and the center of mass $\bar{x}$.



$$M = 20(-1) + 1(5) + 5(9) = -20 + 5 + 45 = 30$$

$$m = 20 + 1 + 5 = 26$$

$$\bar{x} = \frac{M}{m} = \frac{30}{26} = 15/13 = .08829$$

2. The masses $m_1 = 6,\ m_2 = 5,\ m_3 = 1,\ m_4 = 4$ are located at the points $P_1 = (1,-2),\ P_2 = (3,4),\ P_3 = (-3,-7),\ P_4 = (6,-1)$. Find the moments M_x and M_y and the center of mass of the system $(\bar{x}, \bar{y})$.

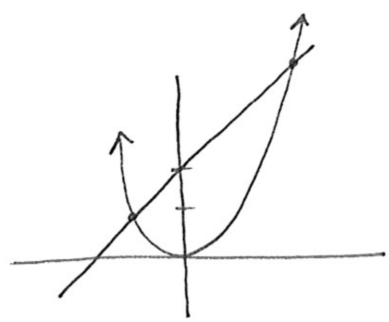
$$m = 6 + 5 + 1 + 4 = 16$$

$$M_y = 6(1) + 5(3) + 1(-3) + 4(6) = 42$$

$$M_x = 6(-2) + 5(4) + 1(-7) + 4(-1) = -3$$

$$\text{Center of mass} = (\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right) = \left(\frac{42}{16}, -3/16\right) = \left(\frac{21}{8}, -\frac{3}{16}\right)$$

3. Find the centroid of the region bounded by $y = x + 2$ and $y = x^2$.



$$\bar{x} = \frac{1}{A}\int_a^b x\left[f(x) - g(x)\right]dx$$

$$= \frac{2}{9}\int_{-1}^{2} x(x+2-x^2)\,dx$$

$$= -\frac{1}{18}x^4 + \frac{2}{27}x^3 + \frac{2}{9}x^2\Big|_{-1}^{2} = 1/2$$

$$\bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}\left[(f(x))^2 - (g(x))^2\right]dx$$

$$= \frac{1}{9}\int_{-1}^{2} (x+2)^2 - x^4\,dx$$

$$= -\frac{2}{45}x^5 + \frac{2}{27}x^3 + \frac{4}{9}x^2 + \frac{8}{9}x\Big|_{-1}^{2} = 16/5$$

$$\text{centroid} = (\bar{x}, \bar{y}) = (1/2, 16/5)$$

$$x^2 = x + 2$$

$$x^2 - x - 2 = 0$$

$$(x-2)(x+1) = 0$$

$$x = -1 \ \& \ 2$$

$$A = \int_{-1}^{2} (x+2) - x^2\,dx$$

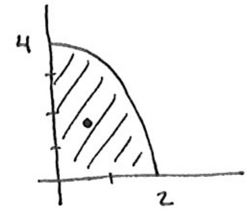
$$= -\frac{x^3}{3} + \frac{1}{2}x^2 + 2x\Big|_{-1}^{2}$$

$$= 9/2$$

4. Consider the region in the first quadrant bounded by $y = 4 - x^2$ and the axes.

a. Sketch the region bounded by the curves, and find the area of this region.

$$A = \int_0^2 4 - x^2 \, dx$$
$$= 4x - \tfrac{1}{3}x^3 \Big|_0^2$$
$$= 16/3$$



$y = 4 - x^2$
OR
$x = \sqrt{4-y}$

b. Find the exact coordinates of the centroid and mark it on the graph.

$$\bar{x} = \frac{1}{A}\int_a^b x\,f(x)\,dx = \frac{3}{16}\int_0^2 x(4-x^2)\,dx = \frac{3}{16}\left[2x^2 - \frac{1}{4}x^4\right]_0^2 = 3/4$$

$$\bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}[f(x)]^2\,dx = \frac{3}{32}\int_0^2 16 - 8x^2 + x^4\,dx = \frac{3}{32}\left[16x - \frac{8}{3}x^3 + \frac{1}{5}x^5\right]_0^2 = 8/5$$

$$\text{Centroid} = (\bar{x}, \bar{y}) = (3/4,\ 8/5)$$

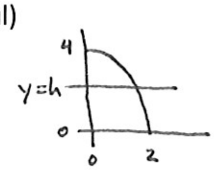
c. Find the number h such that the line $y = h$ cuts the region in part (a) into 2 regions which have equal areas. (hint: integrating with respect to y could be helpful)

way 1:

$$8/3 = \int_0^{\sqrt{4-h}} 4 - x^2 - h\,dx$$
$$8/3 = 4x - \tfrac{1}{3}x^3 - hx \Big|_0^{\sqrt{4-h}} = \left(\tfrac{8}{3} - \tfrac{2h}{3}\right)\sqrt{4-h}$$
$$4 = (4-h)\sqrt{4-h}$$
$$4^{2/3} = 4 - h$$
$$h = 4 - 4^{2/3} = 1.4801579$$

way 2:

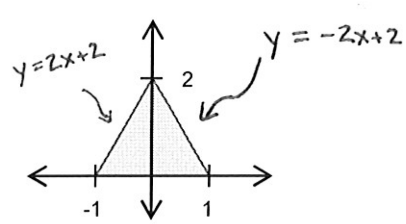
$$\int_0^h \sqrt{4-y}\,dy = \int_h^4 \sqrt{4-y}\,dy$$
$$-\tfrac{2}{3}(4-y)^{3/2}\Big|_0^h = -\tfrac{2}{3}(4-y)^{3/2}\Big|_h^4$$
$$(4-h)^{3/2} - 8 = -(4-h)^{3/2}$$
$$(4-h)^{3/2} = 4$$
$$4 - h = 4^{2/3}$$
$$h = 4 - 4^{2/3} = 1.4801579$$



$y = 4 - x^2$
$x = \sqrt{4-y}$

5. Calculate the moments M_x and M_y and the centroid of the given shape where $\rho = 2$.

$$M_y = \rho\int_a^b x f(x)\,dx$$
$$= 2\int_{-1}^0 x(2x+2)\,dx + 2\int_0^1 x(-2x+2)\,dx$$
$$= \tfrac{4}{3}x^3 + 2x^2\Big|_{-1}^0 + -\tfrac{4}{3}x^3 + 2x^2\Big|_0^1$$
$$= -2/3 + 2/3 = 0$$



$$M_x = \rho\int_a^b \tfrac{1}{2} f^2(x)\,dx$$
$$= \int_{-1}^0 (2x+2)^2\,dx + \int_0^1 (-2x+2)^2\,dx$$
$$= \tfrac{4}{3}(x+1)^3\Big|_{-1}^0 + \tfrac{4}{3}(x-1)^3\Big|_0^1$$
$$= 4/3 + 4/3 = 8/3$$

$$\text{Area} = \tfrac{1}{2}\cdot 2\cdot 2 = 2$$

$$m = \rho A = 4$$

$$\text{Centroid} = (\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right) = (0,\ 2/3)$$