Key

Homework Set 2

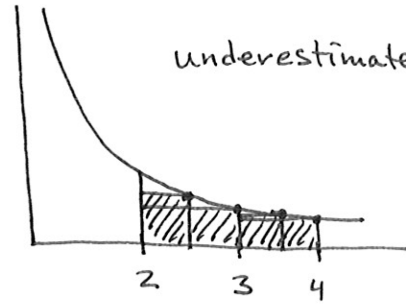
(sect 5.1 & 5.2: Riemann Sums)

Unless otherwise indicated, give your answers correct to 6 decimal places.

1. For both parts (a) and (b), estimate the area under the curve $f(x) = 2/x$ from $x = 2$ to $x = 4$ using 4 approximating rectangles and the indicated endpoints. Sketch the graph and the rectangles. Is your estimate an underestimate or an overestimate?

   a. Right endpoints.

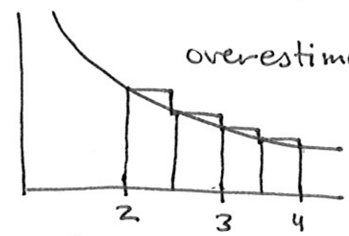
$$R_4 = \Delta x\left[f(2.5)+f(3)+f(3.5)+f(4)\right]$$
$$= \tfrac{1}{2}\left[\tfrac{4}{5}+\tfrac{2}{3}+\tfrac{4}{7}+\tfrac{1}{2}\right]$$
$$= 1.26904761$$



underestimate

   b. Left endpoints.

$$L_4 = \Delta x\left[f(2)+f(2.5)+f(3)+f(3.5)\right]$$
$$= \tfrac{1}{2}\left[1+\tfrac{4}{5}+\tfrac{2}{3}+\tfrac{4}{7}\right]$$
$$= 1.519047619$$

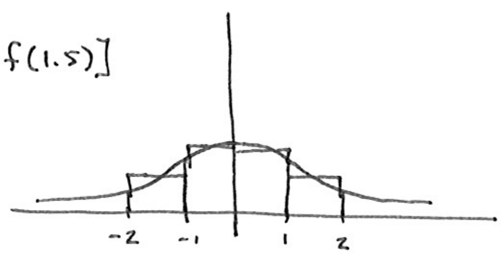


overestimate

2. For both parts (a) and (b), estimate the area under the curve $f(x) = e^{-x^2}$ on $-2 \le x \le 2$ using the indicated number of approximating rectangles and midpoints. Sketch the graph and the rectangles. Does your estimate improve?

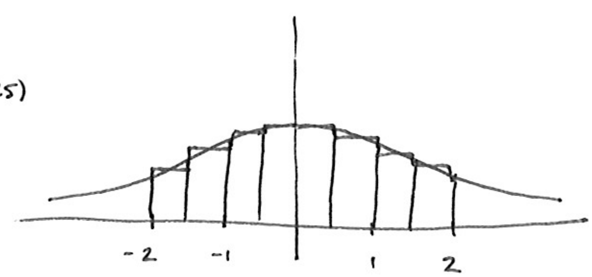
   a. $n = 4$

$$M_4 = \Delta x\left[f(-1.5)+f(-.5)+f(.5)+f(1.5)\right]$$
$$= 1\cdot\left[e^{-9/4}+e^{-1/4}+e^{-1/4}+e^{-9/4}\right]$$
$$= 1.768400015$$



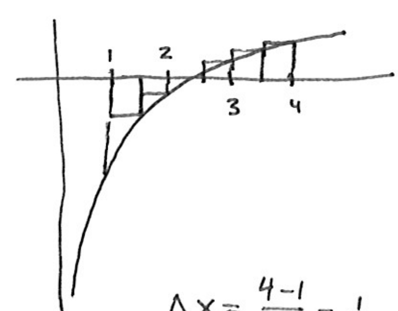
   b. $n = 8$

$$M_8 = \Delta x\left[f(-1.75)+f(-1.25)+f(-.75)+f(-.25) + f(1/4)+f(3/4)+f(5/4)+f(7/4)\right]$$
$$= \tfrac{1}{2}\left[2e^{-49/16}+2e^{-25/16}+2e^{-9/16}+2e^{-1/16}\right]$$
$$= 1.76557789$$



3. If $f(x) = \ln x - 1, 1 \le x \le 4$, evaluate the right Riemann Sum with $n = 6$. What does the Riemann Sum represent? Illustrate with a diagram.

$$R_6 = \Delta x\left[f(1.5)+f(2)+f(2.5) + f(3)+f(3.5)+f(4)\right]$$
$$= \tfrac{1}{2}\left[\ln(3/2)-1+\ln(2)-1+\ln(5/2)-1 + \ln 3 - 1 + \ln(7/2)-1+\ln 4\right]$$
$$= -.12371368$$



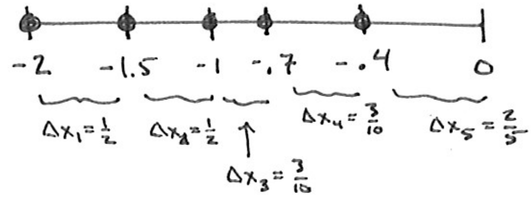
$\Delta x = \frac{4-1}{6} = \frac{1}{2}$

4. Find the Riemann Sum for $f(x) = x + x^2, -2 \le x \le 0$, if the partition points are $-2, -1.5, -1, -0.7, -0.4, 0$ and the sample points are left endpoints. What is the max Δx_i?

$$\text{Sum} = \Delta x_1 f(-2) + \Delta x_2 f(-1.5) + \Delta x_3 f(-1) + \Delta x_4 f(-.7) + \Delta x_5 f(-.4)$$

$$= \tfrac{1}{2}\cdot 2 + \tfrac{1}{2}\cdot\tfrac{3}{4} + \tfrac{3}{10}\cdot 0 + \tfrac{3}{10}(-.21) + \tfrac{2}{5}(-.24)$$

$$= 1.216 \qquad \text{and} \qquad \max \Delta x_i = 1/2$$



5. Consider the curve $f(x) = x^3$ on the interval $0 \le x \le 1$.

a. Find an expression for the area under the curve as the limit of a right Riemann sum.

$$\Delta x = \frac{1-0}{n} = 1/n$$

$$x_i = 0 + i\Delta x = i \cdot \frac{1}{n} = i/n$$

$$R_n = \Delta x\left[f(x_1) + \dots + f(x_n)\right] = \frac{1}{n}\left[\left(\frac{1}{n}\right)^3 + \left(\frac{2}{n}\right)^3 + \dots + \left(\frac{n}{n}\right)^3\right]$$

$$Area = \lim_{n\to\infty} \frac{1}{n}\left[\frac{1^3 + 2^3 + \dots + n^3}{n^3}\right]$$

b. Compute the limit found in part (a). [Hint: $1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$]

$$\lim_{n\to\infty} \frac{1}{n}\left[\frac{1^3 + 2^3 + \dots + n^3}{n^3}\right]$$

$$= \lim_{n\to\infty} \frac{1}{n^4}\left(1^3 + 2^3 + \dots + n^3\right)$$

$$= \lim_{n\to\infty} \frac{1}{n^4}\left[\frac{n(n+1)}{2}\right]^2$$

$$= \lim_{n\to\infty} \frac{n^2(n^2+2n+1)}{4n^4}$$

$$= \lim_{n\to\infty} \frac{n^4 + 2n^3 + n^2}{4n^4}$$

$$= \lim_{n\to\infty} \left(\frac{1}{4} + \frac{1}{2n} + \frac{1}{4n^2}\right)$$

$$= \frac{1}{4}$$

For questions 6 – 8, express the limit as a definite integral on the given interval.

6. $\lim_{n\to\infty}\sum_{i=1}^{n}(x_i - \sin x_i)\Delta x,\ \ [1,5]$

$$\int_1^5 x - \sin x \, dx$$

7. $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{x_i}{1+e^{x_i}}\Delta x,\ \ [0,3]$

$$\int_0^3 \frac{x}{1+e^x}\, dx$$

8. $\lim_{n\to\infty}\sum_{i=1}^{n}\left(1-\frac{i}{n}\right)^4\cdot\frac{1}{n},\ \ [0,1]$

$$\Delta x = \frac{1-0}{n} = \frac{1}{n}$$

$$x_i = 0 + i\Delta x = \frac{i}{n}$$

$$= \int_0^1 (1-x)^4\, dx$$

For questions 9 – 10, express the definite integral as a limit of right Riemann Sums.

9. $\int_{-1}^{2}(1-5x)\,dx$

$$\Delta x = \frac{2-(-1)}{n} = \frac{3}{n}$$

$$x_i = -1 + i\Delta x = -1 + \frac{3i}{n}$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}(1-5x_i)\Delta x$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left(1-5\left(-1+\frac{3i}{n}\right)\right)\cdot\frac{3}{n} = \lim_{n\to\infty}\sum_{i=1}^{n}\left[6-\frac{15i}{n}\right]\cdot\frac{3}{n}$$

10. $\int_3^{10}\frac{\ln x}{x}\,dx$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{\ln x_i}{x_i}\Delta x$$

$$\Delta x = \frac{10-3}{n} = \frac{7}{n}$$

$$x_i = 3 + \frac{7i}{n}$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{\ln\left(3+\frac{7i}{n}\right)}{3+\frac{7i}{n}}\cdot\frac{7}{n}$$